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HERMANN GRASSMANN.

1809-1877.

WE like to believe that the final significance of any thinker's work is independent of his time and place and is fixed by reference to some absolute standard. However that may be, it seems quite clear that his importance in his own age, and hence his effect on the next succeeding generations, depends to some extent on other factors than his intrinsic value. And so in judging that value we must distinguish plainly between it and what we might call the relative or historical importance of the man's work. This latter may well be compared to the potential of a body in electrostatics. For just as that potential depends not only on the actual charge on the body, but also on the charges on neighboring bodies; so also the relative importance of a man is not determined alone by the content of his life and work, but is affected also by his *milieu* and by the reactions of that *milieu* to it.

This is the reason why the contemporary estimate of a thinker is often so utterly wrong. At the time, the external man and his work are more easily seen; but the subtle tendencies of the age are not so readily understood, nor can the observer escape the distortion of vision wrought by prevailing influences on himself. So it comes about that he who is written down a failure in one age may stand out a very genius in the next.

These reflections are quite pertinent to any inquiry into

the life and work of the author of the *Ausdehnungslehre*—Hermann Günther Grassmann, the distinguished mathematician whose own generation passed him by. Although he reached eminence in other branches of human activity, we speak of him as a mathematician because that was certainly the subject he loved most and in which his influence will be most felt in the future. Of him, on the occasion of his centenary (1909), F. Engel[1] could say: "To-day he is known by name to mathematicians, but few have read his writings. Even where his ideas and methods have been diffused in mathematical physics people learn them second-hand, sometimes not even under his name." So in Grassmann we have a straightforward example of a man between whose work and whose influence on his own and immediately succeeding generations we must sharply distinguish if we are to avoid underrating his significance.

He was born on April 15, 1809, in Stettin.[2] His father, Justus Günther Grassmann, was a teacher in the Gymnasium there, and was himself a good mathematician and physicist.[3] His school days passed without his showing any inclination or aptitude in special studies. He had however great skill in and fondness for music, and received a good foundation in piano and counterpoint from the famous composer Loewe. The latter was appointed teacher in the Stettin Gymnasium in 1820 and lived for the first year in the house of the Grassmanns, where he found very congenial society in Hermann and his brothers and sisters, all of whom were musical. With them Loewe often used to try over his new quartettes.

Of Grassman's inner development during these out-

[1] F. Engel, Speech on "Grassmann in Berlin," to the Berliner Mathematische Gesellschaft (1909). To this I owe most of the information about Grassmann's early life given in what follows.

[2] The same date as Euler.

[3] He invented an air-pump cock which was given his name, and also constructed a useful index notation of crystals.

wardly calm and uneventful years we can form a clear picture from his own writings. For in 1831 he wrote an account of his life in Latin in connection with the examination for his teacher's certificate; and later, in 1834, he handed in an autobiography to the Konsistorium in Stettin when he was passing his first theological examination. He refers to those earlier years as a period of slumber, his life being filled for the most part with idle reveries in which he himself occupied the central place. He says that he seemed incapable of mental application, and mentions especially his weakness of memory. He relates that his father used to say he would be contented if his son Hermann would be a gardener or artisan of some kind, provided he took up work he was fitted for and that he pursued it with honor and advantage to his fellow men. As he usually spent his holidays in the country among relatives, and nearly always in the families of clergymen, he conceived the desire to prepare himself for the ministry. But he soon came, partly from the ridicule of his companions and partly from the warnings of his parents, to doubt his capacity. He says however that, after his course of instruction for confirmation, a light came into his dreams. Suddenly he determined to exercise all his intellectual powers and to overcome as far as possible the phlegmatic character of his temperament. And this resolution he carried out with resistless energy.

F. Engel[4] sums up these early years in the following words: "He does not belong to those early ripening geniuses who, even in childhood's years, know whither their gifts will lead them, and turn without doubt or hesitation to that branch of knowledge to which they are called. He was exceptionally gifted on too many sides for that. But even these many-sided gifts by no means showed themselves at the beginning; and that they developed themselves richly

[4] *Ibid.*

later came by no means without effort, but was the direct result of many years of concentrated work which he did in order to develop his character and to solidify his moral outlook and grasp of life."

In the August of 1827 Grassmann and his elder brother Gustav entered the University of Berlin with the intention of studying theology. Two days after their arrival Hermann wrote a droll letter to his mother vividly describing how they had settled in. He tells how they had to climb seventy-two steps to their attic dwelling at 53 Dorotheenstrasse at the corner of Friedrichstrasse. They had only room for their beds and two chairs, but he comments humorously on their extra fine look-out over the gardens and houses of the city, and adds that though the rooms were small they could be the more easily heated. His landlady will be recognized by students all the world over in his pen picture, "If she does talk too much, she is very pleasant and industrious." Particularly amusing is the manner in which he tells how they had spent practically all their money in two days. He enumerates all the possible and impossible things on which they had *not* spent the money, and finally confesses that their sudden impecuniosity was due to the piano which they luckily bought for 50 Taler.

Grassmann admitted later that when he first came to the university he was quite dependent on the guidance of the professors. He was easily impressed by the lectures he heard and tended to fit in his studies with the lectures he chanced upon rather than to take those corresponding to a course of study. At first he came specially under the influence of the well-known church historian Neander. Gradually, however, he became attracted more and more to Schleiermacher to whom he acknowledges great indebtedness. He wrote: "Early in my second year I attended Schleiermacher's lectures, which of course I did not understand; but his sermons began to exercise an influence upon

me. However it was not until my third, and last, year that Schleiermacher entirely engaged my thought, and although at that time I was more occupied with philology, yet I then for the first time recognized how one could learn something from him for every branch of knowledge, because he aimed less at giving positive information than at making us capable of attacking each investigation in the right way and of carrying it on independently." From this we can see how Grassmann was coming to feel the joy of original creative work.

Though he had studied theology with his heart in his subject, he had by this time reached the decision to lay it aside. He says that he had noticed that clergymen who lived in country parishes, shut off from intercourse with scholars, lost grasp of their studies, however enthusiastic they had previously been, and ceased to pursue any investigation on their own account. To escape such a fate he decided to prepare himself as broadly as possible. For this reason he began the study of philology, but he continued it from sheer love of the subject. He had also made the discovery by this time that academic lectures are only of profit if taken in moderation; so he confined himself to two courses under Professor Boeckh, on the history of Greek literature and on Greek antiquities respectively. But he planned out a tremendous course of study, intending to begin with Greek grammar, then to read the Attic authors chiefly the historians, with the study of whom he would combine Greek history and antiquities—next the tragedians with mythology and poetic forms, and afterward Homer and Herodotus. Meanwhile he would seek variety by reading Roman authors. Finally, as he intended to follow his linguistic studies with mathematics, he meant to save Plato and Demosthenes until he began that study.

This exhaustive program he was not able to complete in Berlin. When he had reached the Attic authors he was

taken ill in consequence of over-work. He describes his illness as neither severe nor dangerous, but it compelled him to slow down and to introduce more variety in order to avoid mental strain.

In this way he was led to the study of the sciences, but he showed his growing independence by working free of the schools. He did not attend a single mathematical lecture while a student in Berlin.

We may now see how wide his range of interests was throughout his university career. He seems to have been striving for as broad a foundation as possible, while at the same time he was building up a truly scientific attitude of mind which would enable him successfully to attack any subject he might turn his attention to. It is as though, as Engel says, he knew from the first that it would be necessary in his life to have more than one iron in the fire.

In the autumn of 1830 he returned to Stettin, and late in the following year took an examination for a teacher's certificate before the Scientific Examination Commission in Berlin. It was at this examination that he handed in the Latin autobiography we have previously referred to, and concerning which Köpke, rector of the monastary school of the Grey Friars in Berlin, comments *"Specimen tum propter rerum ubertatem tum propter stili venustatem et elegantiam laude dignum."* He was given permission to teach philology, history, mathematics, German and religious knowledge in lower and middle classes; but the commission at the same time expressed their expectation that he might easily perfect himself for teaching ancient languages and mathematics in all classes. This may have stimulated Grassmann to further mathematical studies, though he had already thrown himself with energy into them under the influence of his father, whose text-books he would naturally use.

He became assistant teacher (*Hilfslehrer*) in the Stettin gymnasium, and in 1832 began to lay the foundations of his great work, the "Theory of Extension" (*Ausdehnungslehre*). He began by working at the geometrical addition of straight lines, or what we now call vector addition. From this he was led to the notion of the geometrical product of straight lines. The direct influence of his father can best be shown by his own words:[5] "But I had not the slightest idea into what a rich and fruitful province I had here arrived; rather did this result[6] appear to me to be little worthy of notice until I combined it with a closely related idea. Namely, by following it up with the same idea of the product in geometry as my father had held,[7] it became evident to me that not only the rectangle but also the parallelogram in general may be considered as the product of two adjoining sides."

He goes on to add that he was surprised to find that he had thus reached a product which changed in sign if its factors were interchanged. And this, together with the fact that he was drawn into other spheres of work—one of which was the passing of the first theological examination at Stettin—caused this seed-idea to remain dormant for some considerable time.

In October, 1834, Grassmann returned to Berlin, this time as mathematics master in a trade school. Soon afterward he applied for a better position than the one he held and his principal gave the following characterization of him: "Mr. Grassmann is a young man not lacking in attainments. It is also apparent that he has given particular attention to the elements of mathematics, and thinks with especial clearness along that line, but he seems to have had

[5] Preface to the first edition of the 1844 *Ausdehnungslehre.*

[6] The notion of writing $AB + BC = AC$ whether the three points A, B, C are in the same straight line or not.

[7] Cf. J. G. Grassmann, *Raumlehre,* Part II, p. 164, and his *Trigonometrie,* p. 10.

little intercourse with people and is therefore backward in the usual forms of social life, shy, easily embarrassed and then very awkward. In the classroom all this vanishes when he does not know that he is observed. He then moves with ease, control, and certainty. In my presence, in spite of the fact that I have done all I could to give him confidence, he has not been able to become fully master of his embarrassment, which caused him much concern. My judgment of him is therefore as yet uncertain, and I cannot say whether he will be able suitably to fill the present vacancy."

As a matter of fact the vacancy was not an easy one to fill, since it had previously been held by no less a person than Jacob Steiner, the geometrician, who had been appointed to the university but retained some of the higher classes in geometry. Grassmann obtained the appointment; and as Steiner had bound himself to initiate his successor as far as possible into his own method of geometrical instruction, one would have expected interesting developments from the contact between the two men. There appears however to have been very little intimacy between them. There was a difference of thirteen years in their ages, and a wide contrast in temperament—the one self-reliant but thoroughly one-sided, the other diffident and many-sided. To these differences in personal characteristics Carl Müsebeck[8] is inclined to attribute their small effect on each other. Victor Schlegel's[9] view was that it was caused by the great difference in the methods employed by the two mathematicians. Whatever may have

[8] Carl Müsebeck, article on Hermann Grassmann, No. 3, Jahrgang 6 of the *Mathematisch-Naturwissenschaftliche Blätter*, p. 1, note.

[9] It is curious to note that V. Schlegel, who, as we shall see, was one of the first appreciators of Grassmann's work, long afterward used the methods of Grassmann's "Geometrical Analysis" to attack the problem of the minimum sum of the distances of a point from given points (*Bull. Amer. Math. Soc.*, Vol. I, 1894, p. 33) and reached a general result which reduces to Steiner's form of solution as a special case; thus illustrating the power of the method.

been the cause it is at any rate clear that Steiner's method of handling geometry had no influence whatever upon Grassmann's manner of thinking.

Several things combined to make Grassmann's stay in Berlin short. He was greatly distressed by the loss of his youngest sister, who was scarcely four years old, and this increased his inclination to religious brooding—to which he was the more inclined as he lacked suitable companionship. His eyesight also gave him some trouble, so that after a year and a quarter he gladly returned to Stettin on January 1, 1836, and became teacher in the Ottoschule.

He had, however, pleasant memories of these months in Berlin, as we can see from a letter written to his brother Robert, in which after speaking with pleasure of his return to Stettin he acknowledges the freedom and mental stimulation afforded by Berlin. At first glance this move from the capital seems a pity, since recognition of his talents might have come to him if he had stayed on. But we must remember to set against this, that he was very high-strung and energetic in mind and could be easily over-stimulated —an effect helped by the quiet life he lived—and also that a calmer atmosphere was more suitable to the long and careful development of his very original way of thought.

While still at the Ottoschule Grassmann entered for and passed the second theological examination in Stettin in July, 1839. We may note here that he was deeply attached to the study of positive theology throughout his life. After passing his theological examinations he became secretary and then president of the "Pomeranian Central Society for the Evangelization of China." And it is noteworthy in this respect that his last work was on "The Falling Away from Belief."

A few months before he submitted his essay for this last theological test, he was examined by the Berlin Scien-

tific Examination Commission in mathematics and physics. It was in connection with this that an event fraught with great consequences to his lifework happened to Grassmann; for he was set the task, by Professor Conrad of the Joachimsthal Gymnasium, of developing the theory of tides. It is uncertain whether the subject was chosen by Conrad on his own initiative or was suggested by Grassmann himself. In any case it was precisely the practical need which was best calculated to spur him on to the development of his dormant mathematical ideas. Later on he spoke[10] of the necessity, in expounding the claims of a new mathematical discipline, of showing its application. And it seems clear that, faced with the difficulties and complications of Laplace's tidal theory, he was led at once to the idea of transforming analytical mechanics by the introduction of his own rudimentary analytical notions. He found to his delight that the new analysis proved a powerful simplifying tool when applied to the equations of Lagrange's *Mécanique analytique*. This initial success encouraged him to extend his method and to clothe many other conceptions such as exponentials, the angle, and the trigonometrical functions, in the form of that analysis. He was then able to simplify and render symmetrical the intricate formulas of the tidal theory. Furthermore he found that the elimination of arbitrary coordinates so effected left the ideas, their development, and their interrelations much less obscured by analytical machinery.

The thesis Grassmann sent to Berlin in April 1840 was of an unusual size;[11] and, in the opinion of Engel,[12] "judged by the number of new thoughts and methods contained in it, there is only one other to be compared with it—the thesis which Weierstrass submitted a year later to the Commis-

[10] In the Preface to the first edition of the *Ausdehnungslehre* of 1844.

[11] It fills 190 pages of royal octavo in the third volume of his *Werke*.

[12] F. Engel, *loc. cit.*

sion at Münster." The two works were, however, accorded very different receptions; and it is evident that Professor Conrad had no idea of the remarkable work he had called into being. His report runs: "The test treats the theory of the tides with thoroughness and strength throughout; and he has chosen, not unhappily, a peculiar method which departs in many particulars from the theory of Laplace." It remains an evil omen for the fate of Grassmann's later work that his examination thesis should thus have failed to find recognition. It must be added that Conrad could scarcely have read the work and still less have been able to estimate it at its true value. For he received it on May 26 and returned it five days later at the oral examination—in which Grassmann fared better, being granted full recognition of his mathematical ability.

Grassmann probably realized that this thesis on tidal theory was but a first fruit of his methods and that those methods themselves were much more general and capable of immense development. This work he threw himself into with characteristic energy in the next few years. He left the Ottoschule at Michaelmas, 1842, and spent six months teaching at the Stettin Gymnasium; after which he entered the Friedrich-Wilhelm-Schule which had been founded a few years before, and of which his eldest son Justus Grassmann is now the principal.

By 1842 Grassmann had completed the main outlines of his new analytical method. He tried to make the ideas known to his own circle by lectures, in which he showed the power of the new "science of extended magnitudes" by further application to mechanics and crystallography. Desiring to expound his method by reference to well-known results he was led to the barycentric calculus of Möbius and to Poncelet. The first of these illustrations was the "Theorie der Zentralen" (Crelle's *Journal*, Vol. XXIV,

1842) in which, without using his own analysis, he made a general statement in which not only all Poncelet's results but also further important general properties of curves and surfaces are contained as special cases. Such wide generalization is characteristic of his method. In 1844 his *Ausdehnungslehre* was published, being designed as the first part of the complete work. This part, which he proposed to follow up with a second later, he called "*Die lineale Ausdehnungslehre,* a new branch of mathematics."

The fate of this book was a tragic one. It remained unread and unsold until the publisher had to get rid of the whole edition as waste paper. Not even a review was granted to it; and what criticism there was had so little basis of understanding that it led to no deeper study of the work. Gauss wrote of it, in 1844, that its tendencies partly went in the same direction in which he himself for almost half a century had wandered; but there seemed to him to be only a partial and distant resemblance in the tendency. He thought it would be necessary to familiarize oneself with the special terminology to get at the real kernel of the book. Grunert declared that he had not completely succeeded in forming a definite and clear opinion about the work. Möbius, whom Grassmann had asked for a review in some critical journal because he stood nearest to the ideas in the book, answered that this mental relationship only existed in regard to mathematics, not with reference to philosophy; and that he considered himself incapable of estimating and appreciating the philosophical element of the excellent work—which lies at the base of all mathematics. But he added that he recognized that, next to the great simplification of method, the principal gain consisted in the fact that by a more general comprehension of fundamental mathematical operations the difficulties of many analytical concepts are removed.

Without entering in detail into a discussion of the causes of this neglect of Grassmann's work[13] we may note that its great generality, its philosophical form, and its original and technical symbolism were contributing factors which also make it very difficult to give any account of the work for the general reader.[14] But the importance of the ideas hidden away in this forbidding volume may be gathered from the words written of it by Carl Müsebeck many years later: "Earlier than Riemann, Grassmann evolved manifolds of *n* dimensions in mathematical analysis. In a lighter and less constrained manner Grassmann arrives by his combinatory multiplication at the fundamental principles of determinant-theory, and the elementary solution of various problems of elimination. In him one finds indicated both Bellavitis's Equipollences and Hamilton's Quaternions." And yet the only recognition given by mathematicians to the ideas of Grassmann was the award to him by the Jablonowski Society at Leipsic for a prize essay[15] on the "Geometrical Calculus of Leibniz" in 1846.

It must not be supposed, however, that Grassmann sat quietly down to neglect. He brought out the importance and applicability of his investigation by numerous valuable articles in Crelle's *Journal,* and later in *Mathematische Annalen* and the *Nachrichten* of the Royal Society of Science of Göttingen. Furthermore, in 1845 he published in Grunert's *Archiv,* Vol. VI, a detailed abstract[16] of the *Ausdehnungslehre*, intended for mathematicians. Thirty years later Grassmann spoke to Delbrück[17] with youthful ardor

[13] See the article below on "The Neglect of the work of H. Grassmann."

[14] An attempt was made to do this by Justus Grassmann in an address delivered at the opening of his school year on April 16, 1909, when the centenary of his father was being celebrated.

[15] *Geometrische Analyse,* published 1847. This treatise is to some extent a substitute for the second part of the *Ausdehnungslehre* of 1844, anticipated in the preface to that work but never written.

[16] Reprinted in the *Werke,* Vol. I, Part I, p. 297.

[17] B. Delbrück, "Hermann Grassmann," Supplement to the *Allgemeine Zeitung,* Oct. 18, 1877.

of this period as one of happy restlessness and joy in discovery. Such joy in original work and faith in the power of his mathematical methods he always retained in spite of a succession of disappointments which would have quenched a less ardent spirit.

It is an extraordinary thing that it was not only in his mathematical work that he failed to find recognition, but also in his contributions to physics. In 1845 he published in Poggendorff's *Annalen* a statement of the mutual interaction of two electric stream lines which was re-discovered thirty-one years later by Clausius. In a school syllabus in 1854 Grassmann stated that the vowels of the human voice owe their character to the presence of certain partial tones of the mouth cavity, a view of the nature of vowel sounds which is usually ascribed to Willis and Helmholtz. Of his other purely physical work we may mention his notes on the mixing of colors and his design of a very simple but practical heliostat.[18] Still he continued to hope that the value of his work would be appreciated. He had himself foreseen[19] that the dislike of mathematicians for a philosophical form might deter them from considering his work, and the comments of Möbius and Grunert on this had shown his fears to be well founded. So he yielded to the often expressed wish of Möbius that he should rewrite the *Ausdehnungslehre* in a form more attractive to mathematicians. In the new work, published in 1862, he chose a more deductive method—one moreover which is not altogether suited to the subject matter, but it did succeed in bringing forward more clearly the original operations and characteristics of the *Ausdehnungslehre*. All was in vain. Neither genius nor indomitable energy could contend against so unresponsive an environment.

We must remember that Grassmann's continued output

[18] A model was constructed by the Stettin Physical Society.

[19] Preface to the first edition of the 1844 *Ausdehnungslehre*.

of virile original work was done in the scanty leisure of an energetic schoolmaster. He had been nominated head-teacher at the Friedrich-Wilhelm-Schule in 1847, and five years later he was appointed successor to his father at the gymnasium. There he remained for a quarter of a century. He had hoped that his mathematical writings would win for him some position in which he would have more leisure for research and be in closer contact with other scientific workers. But it must not be supposed for an instant that this lessened his intense interest in the work at hand. He wrote articles on educational subjects as well as a number of text-books for school use. Of these his *Arithmetik*, written in collaboration with his brother Robert showed a strictness in its proofs which made it a good introduction to the theory of numbers. His *Trigonometrie* has a richness of content in small space and an originality of plan not often then found in elementary hand-books.

Müsebeck has questioned some of Grassmann's pupils on his methods of teaching. They appear in the main to agree with Wandel, who says in his "Studies and Characters from Ancient and Modern Pomerania" that he was a lovable and painstaking master whose kindly instruction was sometimes too difficult for them. The lively interest he took in the independence of those he taught is shown by the fact that, according to Schlegel, he formed a society out of every three scholars in his chemistry class, the members of which had to demonstrate and lecture to the others on some substance and its combinations. The pleasant footing he established between himself and his classes may be judged from the fact that they were willing to co-operate in classwork with him when in later years he had to be taken to school in a wheeled chair. Whenever any of his old pupils speak of him they do so with the greatest admiration and respect.

It is difficult in thus giving an account of Grassmann's educational and scientific activity to avoid at the same time conveying the impression of a mere enthusiastic pedant. It does not seem that there could have been time for anything else. And yet such a view would be widely removed from the truth. For in the midst of all these exacting duties he had many social and general interests. In 1848 he took an active part in politics, expressing anti-revolutionary sympathies; he attempted to introduce a German plant-terminology into botany; and his early developed love for music found expression in organizing an orchestra of his scholars and in collecting numerous folk-songs, which he set for three voices, to be sung in his family.

We have been led, by the necessity of obtaining some idea of the actual conditions under which Grassmann worked, to speak of his later life. We must now return to the time when he first began to realize how slight a recognition was to be accorded to his mathematical writings—that is to say about the year 1852. Great as was his inner sureness of the value of the work, yet his was not the type of mind to be satisfied with a partial success. And so he took the astonishing (and almost unprecedented)[20] step of turning his attention to another field of knowledge altogether and quickly winning the recognition of experts. The pliability of his genius enabled him to force his way into a new subject, philology, and to produce results of outstanding merit in it.

B. Delbrück[21] gives an interesting account of how Grassmann turned to philology. The rules of the traditional school grammar with its mass of exceptions must have been painful to his mathematical understanding, and

[20] The equally neglected English genius Thomas Young combined mathematical and philological ability.

[21] B. Delbrück, *loc. cit.*

so he first planned a grammar and reading book in which scientific laws replaced the old rule-of-thumb methods wherever possible. It is natural therefore that he should next turn his attention to that sphere of language in which such laws are most easily recognizable, namely phonetics. His first attempt in the realm of comparative philology was on this subject. It was an article, published in 1859, on the influence of *v* and *j* on neighboring consonants, and on certain phenomena in connection with aspiration. Delbrück expresses the opinion that his work in this field is not distinguished either for breadth of scholarship, since he worked with few books, or for etymological depth. "But," he says, "it is the clearness of reflection which penetrates into all corners of the subject, the persistence with which the material has been so long accumulated until it became possible to reach the simplest formulation of the governing law, and the untiring nature of the mathematical abstraction which in these undertakings so clearly comes to light."

Grassmann must have quickly recognized how valuable in all researches into comparative philology a deep acquaintance with the oldest Indian languages would be, and he determined with his usual persistency to make himself at home in the hymns of the Vedas. These Sanskrit studies led him to the production of works which rendered his name famous. In 1861 he had only the first volume of the upright text and scarcely half of the Böhtlingk-Roth dictionary. Yet with these means he succeeded in mastering the extraordinary difficulties of the texts, and began his dictionary and translation of the Rig-Veda. He arranged his dictionary in an original manner so as to be able to give the meaning of each form according to the place in which it occurred. Although Delbrück credits the first volume of the dictionary with etymological value for its grammatical subdivision of the roots, yet he regards the arrange-

ment just mentioned as unphilological. It aims less at giving definite historical and philological information than at making successive attempts at explanation. As, however, the work progressed, aided by the stream of material reaching the author from the growing Roth dictionary and elsewhere, it became more philological. Still the method pursued was the same, and Grassmann completed the translation side by side with the dictionary. For long these works formed a useful tool in attacking the difficulties of the Vedas. The recognition of experts was worthily expressed by Rudolph Roth, on whose word the University of Tübingen conferred upon Grassmann the honorary degree of Doctor of Philosophy. He spoke of him as a man *qui acutissima vedicorum carminum interpretatione nomen suum reddidit illustrissimum.*

During this period of his life when he was winning fame in another sphere of work, Grassmann's mathematical writings were gradually obtaining the recognition which was their due. Toward the end of the sixties considerable attention was paid by mathematicians to higher algebra, and the quickening of thought along those lines made recognition much more likely. Hermann Hankel in his *Theorie der complexen Zahlensysteme* of 1867 was the first to call attention to Grassmann's work. Clebsch[22] also shortly afterward accorded him a full measure of admiration. Grassmann[23] believed that Clebsch would have fertilized the theory of extension with far-reaching new ideas of his own if death had not cut short his promising career.

Some of the younger teachers at the Stettin Gymnasium had become pupils of Grassmann; and one of these, the mathematician Victor Schlegel, in his *System der*

[22] Clebsch, *Zum Gedächtniss an Julius Plücker,* 1872.

[23] See preface to second edition of the *Ausdehnungslehre* of 1844, published in 1878.

Raumlehre (1st part 1872, 2d part 1875) made his works more accessible by a clear exposition and application of them. The best kind of approval from authorities came to him in their use of his methods in various fields; and Grassmann himself, after a long interval, again took up his mathematical labors. Of the many articles from his pen, we may mention especially that on the application of his work to mechanics,[24] because it was in this domain that he considered the theory of extension to be particularly successful. He expressed the desire that it might be granted to him to write a treatise on mechanics based on his principles. This was denied him. He lived, however, to see a second edition of his ill-fated *Ausdehnungslehre* of 1844 called for; and died, while it was passing through the press, on September 26, 1877, in his sixty-ninth year. To the last, in spite of great bodily suffering, he retained his vigor and enthusiasm. Five essays published in the year of his death testify to this.

It is a pleasant thing to think that he received such rich recognition before he died; though it must always remain a source of regret that he never succeeded in obtaining the position he hoped for, which would have enabled his powers to be more fully developed and his influence more widely expressed. And yet, there can be no cause for sorrow if we think of the fortitude of this strong soul, and remember the firm conviction expressed in the closing words of the introduction to the *Ausdehnungslehre* of 1862, that his mathematical ideas would some day arise again, though perhaps in a new form, and become part of living thought. To some extent that conviction has proved a justifiable one. The publication of his Collected Works was suggested by Professor Klein of Göttingen. After ob-

[24] "Die Mechanik nach den Principien der *Ausdehnungslehre*," *Math. Annalen*, Vol. XII, 1877.

taining the consent of Grassmann's relatives he laid the matter before the Royal Saxon Academy of Sciences in October, 1892. A committee was formed and F. Engel made chief editor. The first part of the first volume appeared in 1894.

Since then there have been many works on the calculus of extension, but it can scarcely be held that they have done more than make a beginning of the development of the suggestions in Grassmann's work. What has been done has been mainly in the domains of spatial theory and higher algebra; mechanics remains still burdened with traditional coordinate systems. This is the more remarkable since the principle of relativity, with its demand for a generalized dynamics of which ordinary dynamics is a special case, offers such a promising field of application.

There is usually, in the sphere of thought, a rational explanation of apparently irrational facts. A minute influence translated into action by the mass of thinking men may give rise to the spirit of their age; and thus its effects, and the negative effects may be just as great as the positive, carried forward in ever-increasing circles to distant generations. So it has been with whatever lies at the base of the neglect of Hermann Grassmann. There has been bequeathed to us something like an unreasoning distaste for his and similar analytical methods, from which has arisen the need for a definite effort to break the spell of the past. The formation of an "International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics" took its origin from such a need.[25] It may therefore be that a just estimate both of the value and limitations of Grassmann's work will only come by the application of a critical method of wider scope than those of his

[25] P. Molenbrock and Shunkichi Kimura, letter to *Nature*, Oct. 3, 1895.

own period. Indications are indeed not wanting that in the modern theory of transformation-groups[26] lies the criterion for a final judgment.

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[26] Lie and Engel, *Theorie der Transformationsgruppen*, Vol. II, p. 748; M. Abraham and P. Langevin, "Notions géométriques fondamentales," *Encyc. des sciences mathématiques*, Tome IV, Vol. 5, p. 2.

[27] I wish to thank Miss Vinvela Cummin and Mr. R. E. Roper for help in the translation of materials for this sketch.

THE NEGLECT OF THE WORK OF H. GRASSMANN.

IT must not be supposed that the neglect of Hermann Grassmann's mathematical work by his contemporaries is merely an incident of his biography. Its consideration involves a much larger question, because Grassmann's fate was shared by other mathematicians of the period in whose work stress was laid on form rather than content. The distinction between the two may be illustrated by reference to the mathematical treatment of quantity. As soon as analysis had generalized that idea so as to include complex quantities, a mathematics based on formal definitions and of a general character could be developed to include them. The meaning of the propositions of such a calculus need not enter into this study. The propositions would constitute a formal deductive series which could be developed without any reference to content. That Grassmann was a pioneer in the movement which made magnitude subordinate and posterior to a science of form was recognized by Hankel,[1] who says, "It was Grassmann who took up this idea for the first time in a truly philosophical spirit and treated it from a comprehensive point of view." In the Introduction (A) to the *Ausdehnungslehre* of 1844 Grassmann puts the matter thus: "The chief division of all sciences is that into real and formal. The former sciences

[1] *Theorie der complexen Zahlensysteme*, p. 16.

image in thought the existent as independent of thinking, and their truth consists in the agreement of the thought with the existent; the latter sciences on the contrary have for their subject-matter that which has been determined by thought itself, and their truth is shown in the mutual agreement between processes of thought." He goes on to consider mathematics and formal logic as branches of a general science of form, and seeks to dissociate this science from such real sciences as the geometry of actual space, although it must form the basis on which all such are built.

That the neglect accorded to Grassmann had nothing to do with any accident of birth or position is shown by the fact that Leibniz, whose name was famous in both mathematical and philosophical circles, shared the same fate in regard to his *Dissertatio de Arte Combinatoria* and later writings of the same kind, in which he sought to set up a formal symbolical calculus with similar aims. Of Grassmann's contemporaries who worked in the same field, we need mention only George Boole (1815-1864) who failed to obtain anything like a due recognition of his genius; and Sir. W. R. Hamilton whose early papers on quaternions were regarded as mere curiosities. Even when the applications of these generalized formal methods to the founding of a calculus of directed quantities of immediate value to physics had been made, we find the important work of Willard Gibbs waiting for years before it became known and made full use of. If, then, we are to explain the neglect of Grassmann's work we shall have to analyze the causes of the apathy and mistrust with which all such work has been received.

The view held by Carl Müsebeck is that in the almost exclusively philosophical form of representation, *which however was grounded in the whole system,* we have to seek the reason why the contemporaries of Grassmann

drew back in terror from deeper study of his early work. He says[2]: "Such a height of mathematical abstraction in which, with the help of a new calculus, laws are inferred in abstract regions about the mutual dependence of abstract constructions in which not even the character of the spatial is maintained, although at the conclusion of almost every section it is shown how the new method could be used with advantage, was never before known." That this has been a very important factor cannot be doubted. Dislike of the philosophical form of his work was expressed to Grassmann by the few mathematicians who noticed his first *Ausdehnungslehre*. He himself says in the preface to the second edition of this book that he expected the work to find fullest recognition from the more philosophically inclined reader. It is only necessary to refer to the application and extension of his ideas which have come from A. N. Whitehead[3] in England and from G. Peano[4] and C. Burali-Forti[5] in Italy to show how well-founded this forecast was. But the analysis cannot rest there. We must inquire further how this dislike arose.

J. T. Merz[6] in his chapter on "The Development of Mathematical Thought in the 19th Century," inclines to the view that a definite distaste for a philosophical form had set in among German mathematicians as a part of the reaction against the exaggerations of the metaphysical unification of knowledge in the schools of Schelling and Hegel. But mathematicians in modern times have, on the whole, been singularly unaffected by philosophical movements. Furthermore the calculus of extension and allied systems have not fully come into their own even in our own day,

[2] In his memoir of Hermann Grassmann, Stettin, 1877.

[3] *Universal Algebra,* Cambridge, 1898.

[4] *Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann,* Turin, 1888.

[5] *Introduction à la géométrie différentielle, suivant la méthode de H. Grassmann,* Paris, 1897.

[6] *History of European Thought in the 19th Century,* Vol. I, p. 243.

when wide syntheses are eagerly sought. It seems to the present writer that it is in the attitude of the plain anti-metaphysical mathematician that we must seek for the explanation of the want of understanding which leads to mistrust of philosophical form. An immense amount of prejudice barred the way to the full development of a general science of form—prejudice due to non-realization of the purely formal claims of such a calculus.[7] And if we could get at the bottom of this not altogether unreasoning mistrust it might be possible to clear away some of the hindrances to a proper understanding of the fundamental importance of Grassmann's work.

To do this we must push our analysis a step further. What steady cause can have been operating over such a long period which could so affect the attitude of the individual as to create what amounts almost to a general blindness to the importance of a whole body of contributions to thought? I believe that the root of the matter lies in wrong principles of instruction. It may be that this at first sight appears too small an influence to have such consequences; but so did the minute geological influences of the uniformitarians to those who sought for explanations in more dramatic cataclysms. It is as unscientific to neglect the unobtrusive but persistent influences of educational methods on pure thought as it would be to treat of the social conditions of a people without taking into account their mind-development.

We will only give one well-recognized example of the importance of methods of exposition on mathematical history. Merz places Gauss at the head of the critical movement which began the nineteenth century. He adds,[8] however, that it was not to him primarily that the great change

[7] Cf. the article below on "The Geometrical Analysis of Grassmann and its Connection with Leibniz's Characteristic," § 2.

[8] *Op. cit.*, Vol. II, p. 636.

which came over mathematics was due, but to Cauchy. Gauss, while issuing finished and perfect though sometimes irritatingly unintelligible tracts, hated lecturing; in contrast to this Cauchy gained the merit, through his enthusiasm and patience as a teacher, of creating a new school of thought—and earned the gratitude of the greatest intellects, such as Abel, for having pointed out the right road of progress. But it is not so much upon the manner of exposition of original mathematicians themselves that stress must be laid. It has without doubt often happened that writers of great analytical insight have failed to see that it is no more a descent to a common level to seek out and use the best methods of enforcing consideration of their work, than it is to use a printing-press instead of a town crier the more effectively to reach their audience. Grassmann himself, however, did all that was humanly possible in this way, although Jahnke is of the opinion that he was inclined to the belief that even first instruction should be rigorous; and kept back applications until too late. It is rather that teaching methods in general during the nineteenth century have always lagged too far behind discovery. And so they have left the students of one generation, who are the potential original workers of the next, with minds unreceptive to newer and more delicate methods. It might be urged that this would affect equally all branches of mathematics, but I think it can be shown that it is on the reception of such fundamental analytical methods as Grassmann's that its evil influence more particularly falls.

It is quite obvious that the subject must be limited if we are to deal in detail with the suggested effects of inadequate educational methods. So I shall confine myself in what follows to the consideration of the difficulties which beset the path of the teacher who has to explain the ordi-

nary concepts of mechanics; and attempt to show how failure to realize the nature of those difficulties tends to produce an unreceptive attitude to modern analysis. I have chosen this subject for two reasons. Firstly, it seems to me that if the concepts of mechanics were properly treated they would finally appear to the pupil as useful constructions instead of as the dogmatically asserted existents they are still commonly held to be; and so the formal science underlying the real science of mechanics would naturally arise for him as the final result of analysis, and not as the unreal fabric of a philosopher's dream. And secondly, it is the domain to which the various "extensive algebras" have peculiar applicability, as Grassmann himself felt strongly. It is highly significant therefore that it is precisely Grassmann's suggestive applications to mechanics whose neglect is the most noticeable. That this is so is. on my view, because sounder and more philosophical notions of geometrical as opposed to mechanical concepts were already coming into exchange in Grassmann's own day so that geometrical applications were thereby rendered more understandable.

At the very outset of our discussion we are faced with the difficulty that so much difference of opinion exists between teachers of mechanics that many have been forced into the conclusion that, since the enthusiast with an unphilosophical method of his own can yet reap good results, method is unimportant. This, of course, is only partially true. If it were wholly true it would mean an end to all possibility of coordination—an end, in fact, to the claims of education to be a science. To grant that education is an art is not to forego all its claims to be a science. For we must regard all art as applied science "unless we are willing, with the multitude, to consider art as guessing

and aiming well."[9] Beneath the apparent chaos of opinion on the teaching of mechanics there is however some order if one can avoid certain sources of confusion which have led to superficial differences of opinion where nothing deeper exists.

One source of confusion is the absence of a clear idea of the difference in educational theory between an impersonal *principle* and the more personal element—the *method* of applying the principle. This distinction is insisted on by Mr. E. G. A. Holmes,[10] and seems a real one. If once we realize it we can see how it is possible for there to be fairly well accepted scientific principles of teaching at the same time as a wide divergence of method in use by different teachers under differing conditions. And indeed if one looks carefully into much of the polemical writing on mechanics teaching it is seen to be caused less by fundamental differences of principle than by differences of method. It is still more necessary to clear away a second source of unsatisfactory discussion. A superficial glance through the mass of controversial writing on science teaching in recent years would lead one to suppose there was a sharp division of principle between those who believe in a logically ordered course with emphasis on what one may call the instructional method, and those who prefer a looser, more empirical, treatment usually embodying heuristic methods. It would be possible, however, to reconcile many of the combatants if they could be persuaded to see that so direct an opposition is far too simple a statement of the problem, and that each may be partial statements of the real solution. And this becomes possible, I think, if once the disputants grant the importance of the biogenetic or embryonic principle as applied to education—the principle, that is to say,

[9] Reference to Plato, *Philebus*: G. Boole, *The Mathematical Analysis of Logic*, note p. 7.

[10] E. G. A. Holmes, *The Montessori System of Education*, English Board of Education Pamphlet, No. 24, p. 3.

that the development of the individual is a recapitulation of the development of the race. It seems strange that it should be necessary at this stage to call attention to a principle so well known[11] and so much applied, and yet one often has the spectacle of a successful teacher of higher classes urging the claims of logical order against an equally successful empiricist whose experience has been with younger pupils. The truth is, of course, that no one method is applicable to all ages. If the biogenetic law holds, then the natural principle would be to use, in general, modes of teaching a subject similar at each stage to those by which the race has gathered its knowledge of that subject. In mechanics this would mean that a more rigidly logical course would follow empirical experiments and the handling of simple machines.

We will now pass on to our main investigation of the factors which must be taken into account in avoiding the creation of an atmosphere uncongenial to a final abstract analysis. In doing so I will indicate what appear to be the general principles by which one must work in giving to beginners living ideas of the entities of mechanics, and failure to comply with which leads to the production of passively instructed, rather than of irritable and responsive, organisms. The concepts of mechanics are produced from the raw material of experience by the process of abstraction, and a beginner must therefore pass through an experimental stage before he is introduced to the logically defined concepts themselves. In fact he must first use and handle rough ideas and thence be led to build up the more rigidly exact definitions of them for himself. It follows from this that any information we can glean

[11] It is a very remarkable thing that De Morgan in his *Study and Difficulties of Mathematics,* first published in 1831, or 28 years before the *Origin of Species,* should have stated this principle so concisely in the words (p. 186) referring to discussions of first principles: "the progress of nations has exhibited throughout a strong resemblance to that of individuals."

about the actual historical process by which man came to form and use concepts may be of vital importance to a teacher. In mechanics particularly, where the concepts are less obvious than in geometry (the first ideas of force, mass, acceleration and energy, regarded however not as constructions but as real entities, were only developed to any clearness after Galileo—that is at quite a late stage in man's history) any fogginess about their nature and use means endless confusion; and that accounts for most of the difficulties commonly experienced.

It was Locke who first plainly showed how concepts arise from the material of immediate perception. If we think of the flux and confusion of our perceptions—the colors, sounds, smells, sensations of touch, at any instant we find our attention drawn to some more insistent parts of that flux. When these continually recur we use nouns, adjectives and verbs to identify them. Such is the beginning of the formation of concepts. These are regrouped to form other concepts. Thus a wide experience of animals would lead us to group them and to speak, for example, of a class "dog." Once classed we can treat all instances as having the general properties of the class. The practical advantages are obvious. "The intellectual life of man consists almost wholly in his substitution of a conceptual order for the perceptual order in which his experience originally comes," says William James.[12] Once concepts are formed they enable us to handle our immediate experience with greater ease. And by building up more and more complex concepts and tracing the connections between them we create our mathematics and our sciences.

Even animals may form rough concepts.[13] A dog by experience comes to know the difference between "man"

[12] *Some Problems of Philosophy*, p. 51.

[13] This treatment of the origination of concepts is founded largely on that of E. Mach in his chapter on Concepts in the volume *Erkenntnis und Irrtum*.

and other animals. Furthermore if he met a dummy man he would soon find out that the reactions he ordinarily associated with "man" failed to be reproduced, and so would reject that experience for his man-class. In a similar way man must have formed concepts becoming more and more complicated but more firm in outline as his experience became richer. But it is to be noticed that the growth of concepts in a body of experience depends on the number and interest of our observations in the region concerned. For this reason interest in, and consequent familiarization with, simple machines and mechanical toys may well be the child's best introduction to mechanics. Model monoplanes, an old petrol engine from a motor cycle, pumps, a screw, levers, a jack, Hero's turbine model—all these can easily be got at; few young children will show no interest, while many of them will possess already in these days of mechanical toys a considerable knowledge of manipulation. Simple explanations of the working of such apparatus are absorbed with astonishing readiness. In larger schools where there is an engineering workshop this method of introducing young boys to mechanics by way of machinery has been tried with considerable success. Knowledge gets picked up as it were "by contact." The concepts which arise at this stage are necessarily crude—general ideas of force, speed, work and friction; this latter is, of course, one of the first things to notice—not the last to be dealt with as is usually the case. Simple as these considerations are, they are not yet fully appreciated. The London Mathematical Association's *Report on the Teaching of Elementary Mechanics* suggested some time ago that the phrase "Mechanical Advantage" be replaced by "Force-Ratio." For beginners neither of these is intelligible; but they very soon know "how much stronger" a machine makes you. And that conception is quite good enough for them to use.

In introducing simple mechanical concepts to beginners, therefore, the principle to use is that the concepts must arise naturally from experience and not be handed out as definitions. Dictated definitions not founded on sufficient knowledge of facts are flimsy constructions ready to fall at the first breath of difficulty. They do not perform that primary function of concepts of helping one to classify and handle facts, because the facts to be handled are not in the mind when the concept is formulated. "How much stronger a machine makes you" is a phrase which reminds the hearer at once of the assistance it gives him in grouping machines and using them intelligently for different purposes. A note-book definition of "mechanical advantage" is likely to present another arithmetical puzzle instead of serving to remind the learner of the solution of old ones. The principle here advocated was well expressed in the discussion on mechanics teaching at the British Association in 1905 by the president of the section, Professor Forsyth. He said, "What you want to do in the first instance is to accustom the boys to the ordinary relations of bodies and of their properties, and afterward you can attempt to give some definitions which will be more or less accurate; but do not begin with the definitions, begin with the things themselves." And the philosophical basis for the principle is, that the significance of concepts is always learned from their relations to perceptual particulars, their utility depending on the power they give us of coordinating perceptual facts. From this it follows further that concepts and names should never be introduced where there is no direct and immediate gain in so doing. Such terms as "centrifugal" and "centripetal" forces, and the endless discussion to which they lead, are thus beside the mark. "Force toward, or away from, the center" does all that is necessary without introducing new words of really less precision.

It should be noted that some of the crude concepts arrived at in the early stages are really, when one comes to analyze them, very complex, and Ostwald's warning[14] against the error of supposing that the less simple concepts have always been reached by compounding simple ones has application here. As he says, complex concepts often in origin have existed first. We can now see more clearly why the teacher of mechanics so often complains of the difficulty of giving the average child a satisfactory notion of force.[15] The difficulty is largely due to the teacher who knows the concept to be complicated, and seeks to define it in terms of mass-acceleration—thus involving two more concepts, one of which (mass) is at least as difficult to understand as force. A rough idea of force, considered simply as a "push" or a "pull," can be assimilated at a very early stage; that of mass-acceleration must come very much later.

The bearing of this preliminary stage in the formation of concepts on our main thesis may now be traced. It is quite evident that the individual has very limited powers of absorbing the logically ordered account of a science in which stress is laid on abstract notions before such notions have grown up naturally by use. Now this difficult step for the beginner from the perceptual to the conceptual is very similar to that which leads from ordinary mechanics to such a treatment of the subject as that of Grassmann. Both lead into regions of greater abstraction. In the latter case we can get rid of concepts in so far as they relate to the existent, and reach a statement of mechanical principles in terms of a generalized form-theory. We may illustrate, roughly, the meaning of this by the following analogue. At different stages in the history of physics various the-

[14] Ostwald, *Natural Philosophy*, p. 20.
[15] Cf. C. Godfrey, Brit. Association *Report on Mechnics Teaching*, p. 41.

ories of light have been held. The concepts used in these theories (corpuscle, elastic-solid ether, electro-magnetic medium) have possessed widely different "qualities"; but the equations expressing the relation between the conceptual elements have throughout possessed similarity of form. A science of form would hence lay emphasis on the invariant relations, refine away the particular concepts, and leave a much more abstract and generalized science.

But if racial development is in the main similar to the progress of the individual this will explain the great difficulty experienced by whole generations of mathematicians in understanding work of the type of Grassmann's.

Furthermore, it is at this point that defective scientific training looms into importance. For unless great care has been taken in avoiding the too early definition of concepts, a rigid view of them is promulgated. The older dogmatic and orderly methods of teaching tended inevitably to this. The consequence was that when the time came for polishing and development of the concepts obtained, and for the deliberate building up of more complex ones—it was found that the capacity for subtle generalized views had been destroyed. A mind forced into passivity and filled with inert knowledge cannot suddenly be brought to discard it in response to the stimulus of a tentative generalization. To take a simple example, the idea of a new kind of addition, applicable to vectors, shocks and confuses a pupil who has been dogmatically instructed in algebra as though it were a sacred rite. As with the child under such a system, so with the generation of which he forms a part. Jahnke states that many mathematicians were put off by meeting in Grassmann's work a product which equals zero without either factor doing so. Formal logical development often leads to conclusions which are not capable of any mental image.[16] Such abstractions are

[16] Cf. F. Klein, *The Evanston Colloquium,* Lecture 6.

out of reach of those who have never been freed from the confines of the existent world.

Cajori[17] in a notice in 1874 of the publication called *The Analyst,* Des Moines, Iowa, said that it bore evidence of an approaching departure from antiquated views and methods, of a tendency among teachers to look into the history and philosophy of mathematics. My thesis is that such a movement, which certainly has not yet been realized, would remove the main cause of the neglect of Hermann Grassmann's work, which even in these days is often granted the kind of recognition accorded to certain literary classics, which are famous but never read. Perhaps it is an earnest of the future that the copy of *The Analyst* referred to by Cajori contained a brief account[18] of the essential features of Grassmann's *Ausdehnungslehre.*

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[17] *Teaching and History of Mathematics in the United States.*

[18] Translated by W. W. Beman of the University of Michigan.

THE GEOMETRICAL ANALYSIS OF GRASSMANN AND ITS CONNECTION WITH LEIBNIZ'S CHARACTERISTIC.

§ 1.

BY a curious turn of fate Grassmann wrote, in the introduction to his "Geometrical Analysis," concerning Leibniz's early work on the same subject, words which were to apply with prophetic force to his own *Ausdehnungslehre*. "When the special power of a genius. . . . is so revealed that he is able to grasp and extend the ideas toward which the development of his time is directed, and so appears representative of his period, then that power shows itself still more remarkable when it can seize ideas in those realms of thought in advance of their day and forecast for hundreds of years the line of their development. While ideas of the first kind are often developed simultaneously by the outstanding spirits of the age, as when both Leibniz and Newton founded the differential calculus—a certain stage of fruition being reached—ideas of the latter kind appeared as the special characteristic of the individual, the innermost revelations of his mind into which only a few elect contemporaries could enter and have a foreshadowing of the developments which were to spread from them in the future. While the first received great applause and aroused movement in their own day,

because they represent the summit of the epoch, the others for the most part fall without effect in the contemporary period since they are only understood by a few, and then only partially. Often afterward does such thought become the seed of a rich harvest. That this great idea of Leibniz—namely, the idea of a true geometrical analysis—belongs to this preparatory and, as it were, prophetic class cannot be doubted for a moment. It has also shared the fate of such. Indeed by a special ill favor of circumstances it has remained hidden far beyond the time when it might have had a powerful influence. For even before it was brought out of its hiding place by Uylenbroek, paths toward a similar analysis had been made in other ways."

At the time when these words were written Grassmann could have had no idea of the disappointment which was to come to him in the neglect of his own work. The first edition of the *Ausdehnungslehre,* or theory of extended magnitudes, had been published in 1844 and had received no attention from mathematicians with the exception of a few individuals. Grassmann, however, believed that recognition was only a matter of time and sought to bring out the importance and applicability of the new analysis. For the year 1845 (but extended to 1846 to coincide with the two hundredth anniversary of Leibniz's birth) the Jablonowski Society of Leipsic set a prize essay demanding the restatement or further development of the geometrical calculus discovered by Leibniz or the setting up of a similar calculus; and the award was made to Grassmann for the essay, printed in 1847, from which I made the above quotation. This was the first and the only acknowledgment of the value of his work which he received from mathematicians until long after many of the ideas he formulated had been reached and applied by other methods and other thinkers.

I have laid stress on the similarity of treatment meted

out to the fundamentally important work of the two men because I believe that in some elements of its explanation lies the clue to unravel the difficulties of their subject matter and connection with each other. The more general aims of both Leibniz and Grassmann were the same—the setting up of a convenient calculus or art of manipulating signs by fixed rules, and of deducing therefrom true propositions for the things represented by the signs, for use as a generalized mathematics. In each case their geometrical calculus was a particular application to geometry of a wider calculus for which each desired more than mere applicability to mathematics.

In a letter to Arnauld, dated January 14, 1688, Leibniz writes[1]: "Some day, if I find leisure, I hope to write out my meditations upon the general characteristic or method of universal calculus, which should be of service in the other sciences as well as in mathematics. I have definitions, axioms, and very remarkable theorems and problems in regard to coincidence, determination, similitude, relation in general, power or cause, and substance, and everywhere I advance with symbols in a precise and strict manner as in algebra. I have made some applications of it in jurisprudence." Similarly Grassmann[2] says: "By a general science of symbols (*Formenlehre*) we understand that body of truths which apply alike to every branch of mathematics, and which presuppose only the universal concepts of similarity and difference, connection and disjunction." The symbols are made so general as to be applicable to both logic[3] and mathematics, although in the *Ausdeh-*

[1] George R. Montgomery (trans.), *Leibniz: Discourse on Metaphysics, Correspondence with Arnauld, and Monadology*, p. 241.

[2] *Ausdehnungslehre* of 1844, p. 2.

[3] The application of such a general science of symbols to formal logic was made by both H. Grassmann and his brother Robert.

nungslehre they are only applied to the domain of mathematics.[4]

It is clear that both Leibniz and Grassmann, but especially the former, claimed great scope for their calculus, a fact which tended to make their writings generalized and difficult to understand. In the preface to his *Universal Algebra* (1898) Professor Whitehead expresses his belief that lack of unity in presentation (which of course would be the tendency in dealing with a method applicable to many fields) discourages attention to such a subject. But that is not all. A new mathematical method, to make itself known, has to appeal in the main to mathematicians and not to philosophers. So that a wide and philosophical treatment is apt to be discounted by the ordinary man who thinks logic can be made to prove anything.

§ 2.

Before we condemn this attitude we must first of all inquire as to what exactly the common man means by the dangers of logic. What he really fears is not logic but fallacy. Without realizing it he distrusts a mechanical dexterity in reasoning because the attainment of truth depends not only on a facility in manipulating logical processes but also on the sifting of first principles. When Leibniz claims for his *characteristica universalis* or "universal mathematics," the germ of which he produced in his *De arte combinatoria* published when he was twenty, that ".... there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other (with a

[4] In the *Ausdehnungslehre,* however, are expressions directly applicable to logic, e. g., there is the generalized expression for the result of division $C+O/B$ where O/B is an indefinite form (p. 213)—an anticipation of Boole's use of O/O to symbolize perfect indefiniteness, as pointed out by Venn in his *Symbolic Logic,* note p. 268 (2d ed.).

friend as witness, if they liked): Let us calculate"—he is running counter to the plain man's knowledge that there are two parts of a logical process, the first the choosing of an assumption, the second the arguing upon it.

Now Leibniz realized of course that premises are required first, but he thought they could be obtained very simply. By analyzing any notion until it was simple he thought that all axioms or assumptions followed as identical propositions. Thus he was led, by his view of ideas, to believe that even the axioms of Euclid could be proved. So in his *New Essays*, "I would have people seek even the demonstration of the axioms of Euclid.... And when I am asked the means of knowing and examining innate principles, I reply....we must try to reduce them to first principles, i. e., to axioms which are identical, or immediate by means of definitions which are nothing but a distinct exposition of ideas." This is connected with his view that all our ideas are composed of a very small number of simple ideas, which together form an alphabet of human thoughts. But, as Couturat remarks,[5] there are many more simple ideas than Leibniz believed; and furthermore there is no great *philosophical* interest in such. "An idea which can be defined or a proposition which can be proved, is only of subordinate philosophical interest."[6] It is precisely the business of philosophy to deal with the primitive, intuitive assumptions on which any calculus must be based.

So the plain man is to some extent justified in his mistrust of the uncritical application of a calculus.

§ 3.

It is very necessary, however, to see exactly what is, and what is not, here granted to the plain man. It is true that in using a calculus we must be careful not to over-empha-

[5] L. Couturat, *La logique de Leibniz*, p. 431.
[6] B. Russell, *The Philosophy of Leibniz*, p. 431.

size the results at the risk of forgetting the premises from which they have been obtained. But that being admitted, thus making the final development of the universal characteristic a matter not of philosophy but of a sort of generalized mathematics of which formal logic[7] and geometry are special cases, it does not follow that there must be limits to the applicability of the calculus in these spheres. Yet that is what the modern representative of our plain man asserts. His criticism of a logical calculus has put on a more philosophical form, but remains essentially the same. Henri Poincaré may justly, I think, be taken as such a representative. For he says, "I appeal only to unprejudiced people of common sense. . . . they [the logisticians] have shown that mathematics is entirely reducible to logic, and that intuition plays no part in it whatever."[8] This belief led Poincaré to the view that, since he knew from his own experience as a mathematician of great insight the important part intuition plays in mathematical discovery, therefore the *nature* of mathematics cannot be logical.

This reasoning is founded on a very common fallacy which I will call the genetic error—the error, namely, which lies in the assumption that the origin of a thing in some way determines its nature.[9] If this assumption is made it follows that since intuition plays a part in dis-

[7] Leibniz himself foresaw this development carried out by Boole, Peano, Frege, Whitehead and Russell and their school of symbolic logicians. In fact he made discoveries in this field but did not publish them because they contradicted certain points in the traditional doctrine of the syllogism. In some points he even advanced beyond Boole (See Couturat, *op. cit.*, p. 386).

[8] *Science et Méthode*, p. 155; cf. also C. J. Keyser in *Bull. Amer. Math. Soc.*, Jan. 1907, pp. 197, 198.

[9] This error has been very common in philosophy. It underlies much argument against rationalism, denying that knowledge reached empirically can be anything other than empirical. (Cf. Leibniz, *New Essays*, IV, 1 § 9, against Locke.) It is at the basis of many criticisms leveled against any generalization of number, since the idea of number arose from perceptual experience. It vitiates pragmatism, which inquires into the causes of our judging things to be true in order to get at the nature of truth. (See B. Russell, *Philosophical Essays*, p. 110.)

covery, the nature of mathematics cannot be purely formal, and therefore it cannot be expressed in terms of symbolic logic. Now all such references to the origins of mathematics are irrelevant. Once the premises have been made, and that is where intuition comes in, symbolic logic is merely "an instrument for economizing the exertion of intelligence."[10] The mind, being relieved of unnecessary work by a good symbolism, is set free to attack more difficult problems; for as Professor Whitehead says,[11] "Operations of thought are like cavalry charges in a battle—they are strictly limited in number." Nor is that the only advantage of this modern development of Leibniz's universal mathematics. It "has the effect of enlarging our abstract imagination and providing an infinite number of possible hypotheses to be applied in the analysis of any complex fact."[12] And so it lends itself to the production of just such novel fundamental hypotheses as are needed in subjects like the dynamics of relativity.

So finally, we must say of the symbols of a universal calculus what Hobbes said of words, "They are wise men's counters, they do but reckon by them; but they are the money of fools." Yet it must be recognized that when it is confined to dealing with mathematics in its widest sense (taken to include formal logic),—within the limits imposed on his own calculus by Grassmann, in fact,—it serves as a powerful and legitimate tool.

§ 4.

This discussion of the neglect and mistrust of mathematicians for the generalized calculus of both Leibniz and Grassmann has, I hope, shown what the nature of such a

[10] W. E. Johnson in *Mind*, N. S., Vol. I, pp. 3, 5. Cf. Stout, "Thought and Language," *Mind*, April, 1891.

[11] *An Introduction to Mathematics*, Home Univ. Library, p. 59. See also P. E. B. Jourdain in *The Monist*, Jan. 1914, p. 141.

[12] B. Russell, *Our Knowledge of the External World*, pp. 58, 242.

calculus is. Moreover, it accounts for the long period which elapsed before their fruitful application of these methods of calculation to special fields obtained the notice they deserved.

The particular application we are here concerned with is that to geometry. In a letter to Huygens of September 8, 1679, Leibniz complained that he was not satisfied with the algebraic methods, and adds: "I believe that we must have still another properly linear geometrical analysis, which directly expresses *situm* as algebra expresses *magnitudinem*. And I believe I have the means for it, and that one could represent figures and even machines and movements in symbols, as algebra represents number or magnitude; I am sending you an essay which seems to me notable." This essay contained an account of his geometrical calculus in which the relative position of points is denoted by simple symbols and fixed without the help of the magnitude of lines and angles. It differs therefore from ordinary algebraic analytical geometry. The further development of this calculus was the subject of Grassmann's *Geometrische Analyse*[13] which we have already noted as being crowned by the Jablonowski Society. This was done on the recommendation of Möbius, who found in Grassmann's essay a generalization and extension of his own barycentric calculus.

We will now consider the geometrical calculus of Leibniz with a view to discovering if Grassmann's development of it has fulfilled in any way Leibniz's hopes of its ultimate importance.

§ 5.

The letters and papers of Leibniz in which he deals with his project of a geometrical calculus are many, and

[13] This treatise develops some of the subjects which Grassmann had intended for a second part of the 1844 *Ausdehnungslehre*, which was never written.

spread over a considerable period of time.[14] The most important is the *Characteristica geometrica,* a sketch of the notion which he made for fear it should be lost if he found himself unable to develop it. The essay enclosed in the letter to Huygens in 1679 was an extract from this. From these writings it seems clear that the starting point was his conviction of the imperfection of algebra as the logical instrument of geometry. Thus, "Algebra itself is not the true characteristic of geometry, but quite another must be found, which I am certain would be more useful than algebra for the use of geometry in the mechanical sciences. And I wonder that this has hitherto been remarked by no one. For almost all men hold algebra to be the true mathematical art of discovery, and as long as they labor under this prejudice, they will never find the true characters of other sciences." It must be noted that algebra is here used by Leibniz in its ordinary sense, not as a general term for any calculus.

He saw that analytical geometry only expressed geometrical facts in a complicated and roundabout manner. A figure such as the circle is not defined by its internal relations, but by reference to its relations to arbitrary coordinates. So a set of magnitudes foreign to the figure are introduced and obscure the purely geometrical relationships. Further, to reduce relations of position to relations of size presupposes Thales's theorem about similar triangles and the theorem of Pythagoras.[15] In other words analytical geometry is thus made dependent on synthetic. The analysis not being pushed far enough, it has not the logical perfection which belongs to a purely rational science.[16] He realized the want of rigor and generality of

[14] An interesting bibliography of them together with an account of the main ideas which inspired and directed his search for a geometrical characteristic is given in Couturat, *La logique de Leibniz,* 1901.

[15] *Characteristica geometrica,* § 5.

[16] Cf. his letter to Bodenhausen.

intuitive methods, but dreamed of a method which would be completely analytical and rational while still possessing all the advantages of a synthetic method.

In this his aim was similar to that which he had in mind for his universal characteristic, which was to be a logical calculus replacing concepts by combinations of signs, and which furthermore was not merely to furnish demonstrations of propositions but to be the means of discovering new ones. So, in like manner, his geometrical calculus was to combine analysis with guidance of the intuition.[17] A fusion of analysis and synthesis being made, the divorce between calculation and construction would disappear. "This new characteristic....will not fail to give at the same time the solution, construction, and geometrical demonstration, the whole in a natural manner and by an analysis."[18] It is clear that the final goal was a science of form of very wide application.[19] This aim we must distinguish carefully from the manner in which he attempted to realize it.

As Grassmann points out in the introduction to his "Geometrical Analysis," this distinction between the distant goal and his attempt toward a new characteristic which he connects with it to render the thought more realizable, is recognized fully by Leibniz. Although the characteristic he provided will be seen to be only a small first step toward the goal he had set himself, yet he had estimated the essential advantages of a final geometrical analysis to an extraordinary completeness. Grassmann says: "Just this eminent talent of Leibniz of being able to foresee in presentiment a whole series of developments without being able to work it out and without dismembering and

[17] Leibniz conjectured that the ancients had some natural and spontaneous analysis of this kind resting on the abstract relations of figures, which underlay and helped their synthetic methods. (*De analysi situs.*)

[18] Letter to Huygens.

[19] *Ibid.* "I believe that one could handle mechanics by these means almost like geometry."

dissecting it, yet to make it present to himself with prophetic mind and to recognize the importance of its consequences—it is just this talent which led him to such great discoveries in almost all domains of knowledge."

§ 6.

Leibniz founds his fundamental definitions on congruence, which means the possibility of coincidence. He represents points whose positions are known by the first letters of the alphabet, and those which are unknown or variable by the last letters. Any two combinations of corresponding points are said to be congruent if both can be brought to coincide without the mutual position of the points being changed in either of the two combinations; so that every point of one combination covers a corresponding point in the other. Congruence (geometrical equality) is a union of two relations—similarity and equality (quantitative equivalence).

All *points* are equal and similar, so all points are congruent.[20] Hence if we use $\equiv$ for congruence, the expression $a \equiv x$, where a is fixed and x is variable, is a definition of space.

It must be noticed that in defining figures by congruence the *axiom of congruence* or *free mobility*[21] must be postulated. If we do this, $ax \equiv bc$ represents a sphere of center a and radius bc.

Also, $ax \equiv bx$ represents a plane which bisects ab perpendicularly.

The above can be taken as the definition of the sphere and the plane respectively. Again $ax \equiv bx \equiv cx$ gives the locus of the center of all spheres which pass through a, b, c; and so it is a straight line.

If $ax \equiv ac$ and $bx \equiv bc$

they together give the common trace of two spheres.

[20] *Characteristica geometrica.*

[21] See B. Russell, *Foundations of Geometry,* Cambridge, 1897.

Combined they are written $abx \equiv abc$. This therefore represents the locus of points whose distances from the points *a, b* are the same as the distances of *c* from *a, b*. That is, it is a circle.

The economical nature of the symbolism is shown by the fact that if we take this as a definition of the circle, it does not imply the idea of the straight line or the plane; nor does it require (as the circle defined by an algebraic equation) that the center of the circle must be known.

As an example of a proof consider the proposition that *the intersection of two planes is a straight line.*

Let $ay \equiv by$ be one plane
and $ay \equiv cy$ be the other.
Then $ay \equiv by \equiv cy$, and this we saw above to be the form of the congruences representing a straight line.

In these examples is a faint foreshadowing of the side by side development of construction, proof and analysis. And since all kinds of spatial relationships can be developed from the line and the sphere, the method is capable of wide extension.

§ 7.

There are several obvious defects in it, however. These appear at once if we attempt by means of it to solve the fundamental problem in geometry of finding the expression for a straight line passing through two given points. Leibniz had previously attacked the problem only to find himself involved in difficulties.[22]

Grassmann's treatment is as follows: We saw above the expression for a straight line was

$$ax \equiv bx \equiv cx.$$

If we now take three auxiliary points, a', b', c', which are not in a straight line, and write

[22] Couturat gives a clear account of this, *op. cit.*, pp. 420-427.

$$\begin{cases} a'x \equiv b'x \equiv c'x \\ a'a \equiv b'a \equiv c'a \\ a'b \equiv b'b \equiv c'b, \end{cases}$$

then together these congruences represent the required straight line through a, b, as the locus of x.
Combining the last two we get

$$\begin{cases} a'x \equiv b'x \equiv c'x \\ aba' \equiv abb' \equiv abc'. \end{cases}$$

This then expresses that the auxiliary points lie on the circle the plane of which is cut at its center by the line ab at right angles.

If this expression is to have the necessary simplicity, it must be possible to eliminate the arbitrary auxiliary points which have nothing to do with the nature of the problem, and to combine the group of formulas into one. That being impossible, the characteristic has failed to serve its purpose.

Indeed the failure of the method followed at once from the choice of congruence as the fundamental relation. For, as we have seen, this complex relation contains a quantitative element, and so prevents any freeing of geometry from considerations of magnitude. In fact, as the above expression for the line through ab shows, we are still left with arbitrary coordinates. Further, in this system there is also ambiguity, as Couturat has shown.[23] In other words the analysis had not gone far enough. If what remained of magnitude had been eliminated—not merely by taking the relation of similarity, for Leibniz had himself shown that to imply metrical relations[24]—but by reducing figures to their projective properties and relations, at least a real geometry of position[25] would have followed. But such a projective geometry, while satisfying Leibniz's desire to

[23] *Ibid.*, p. 428, note 2.
[24] *"Elementa Nova Matheseos Universalis."*
[25] Developed by Staudt, *Geometrie der Lage*, 1847.

eliminate algebraic methods from geometry, would not have been a geometrical calculus with points as elements. Nor could it have had the wide application which he sought for in his calculus; for if it was to be applicable to mechanics and physics, it must at some point have been susceptible of metrical development.

Now, throughout our discussion we have seen that Leibniz was seeking for a characteristic particularly applicable to geometry but akin to his universal characteristic. At the end of the letter to Huygens he says: "I believe it is possible to extend the characteristic to things which are not subject to imagination." In other words he was seeking a formal calculus, an abstract mathematics lying at the base of geometry and applicable not only to it but also to logic. Now Grassmann had already developed such a science of form in his *Ausdehnungslehre* of 1844. So when the Jablonowski Society announced the subject of their prize essay he took the opportunity of expounding his science of extensive magnitudes, not as he had originally derived it, but starting from Leibniz's characteristic.

§ 8.

When he had proved the insufficiency of the relation of congruence as Leibniz had left it, he tried to give it a form in which substitution would be possible. What are congruent to the same thing are congruent to each other, but that does not mean that we can in a general way place instead of a given term in a congruent expression one congruent to it. So substitution is not possible. This can be seen at once. All points are congruent. Therefore, if one could substitute the congruent, one could place *abc* congruent to every combination of three points—which is absurd.

Grassmann rightly regarded the fact that substitution was not possible as a serious defect in the calculus. So he

inquired what equations would hold between the points a, b, c, d, e, f, if $abc \equiv def$.

There must be some function f such that, when the above holds,

$$f(a, b, c) \equiv f(d, e, f).$$

So he was led to the general linear relation of collinearity.

Now in the *Ausdehnungslehre* Grassmann had reached the fruitful idea of a true geometrical multiplication which has the peculiarity that if any two factors of the product are interchanged the sign of the product is changed, that is,

$$\mathrm{AB} = -\mathrm{BA}.$$

This *combinatory* multiplication enabled him now to give an intrinsic definition of geometrical figures in terms of points, and so to accomplish what Leibniz had failed to do. Thus the product ab determines the straight line between the points a, b; the product of three points determines the plane, and so on. But since the product is non-commutative these figures when so defined have a sense represented by the signs + or —. Furthermore, he conceived the notion of using these products to express not only relations of position but also of magnitude. So that the same analysis which gave a geometry of position also gave, side by side and without confusion, a metrical geometry. In making this step he had to define (§ 3, *Geometrische Analyse*) what he meant by a point magnitude. Each element (point, line, plane) has two aspects—its position in space, and its intensity. In the case of the point, this latter was represented by a positive or negative "mass."

By now defining a *line magnitude* as the combination ab of the point magnitudes a, b— the direction of which is through a and b, and the intensity of which can be defined; and also defining the *point magnitude* as the combination AB of two line magnitudes, the position of which is the intersecting point of A and B and the mass value of which

can be made the subject of a definition—then by an assumption which makes $ab = 0$ and $AB = 0$ represent coincident lines and points, it is possible to write in the form of an equation every linear dependence.

Thus $(ab)(cd)e = 0$ denotes that e is the intersecting point of ab and cd.

So the principle of collineation can be expressed, though cumbrously without further adaptation, by such combination equations.

In this way *equality* is made to include the two relations of identity of position and equality of intensity. So projective and metrical relations can be expressed in one form, and considered either separately or together.

§ 9.

It is impossible to follow Grassmann's development[26] further without setting up a technical symbolism, but it may easily be shown how brilliantly Leibniz's hopes of an analysis specially applicable to mechanics have been fulfilled.

In terms of this calculus the sum of n points is their mean point. If intensities are considered, the metrical relation follows. Thus if the intensities represent masses at the points the sum gives the center of gravity of the system—a point whose intensity will be the sum of the other intensities. If the intensities represent parallel forces acting at the point the sum gives the point of application of the resultant. The *barycentric calculus* of Möbius is thus included in this more general analysis.

Furthermore, the line magnitude of Grassmann expresses a force with exactitude. Composition of forces thus becomes the addition of line magnitudes. The general equations of dynamics can also be represented (§ 11, *Geo-*

[26] Needless to say the above is a mere sketch of the beginning of Grassmann's "Analysis." In particular no mention is made of his distinction between *inner* and *outer* products.

metrische Analyse) by means of this calculus, as soon as certain modes of treating infinitesimals have been evolved.

Moreover the possibility of attaching a metrical coefficient to each point in space opens at once many fields of application in physics.

We must notice in addition that the "Geometrical Analysis" does not treat of the *quotients* of non-parallel stretches, a subject which leads to a calculus for dealing with powers, roots, logarithms and angles.

Grassmann can claim justly therefore, as he does, in the concluding remarks to this work, that his mode of treatment, if transferred to physics in general, would simplify the mathematical treatment in a splendid manner. He himself has shown the great advantages of the calculus in many fields. In the essay we have several times referred to, Leibniz wrote, "If it [the characteristic] were set up in the manner I conceive, one could construct in symbols, which would only be the letters of the alphabet, the description of any machine.... One could by these means make exact descriptions of natural objects."

As an example of such descriptive power Grassmann mentions his application of the calculus to crystallography (cf. *Ausdehnungslehre* of 1844, § 171).

§ 10.

Apart from the adaptability of the geometrical calculus to different provinces, there are other good reasons for believing that it realizes the ideal toward which Leibniz looked forward.[27] Grassmann's claim put forward in his concluding remarks will, I think, be granted by any one willing to master the symbolism sufficiently to under-

[27] Letter to Huygens: "Algebra is nothing but the characteristic of indeterminate numbers, or of magnitude. But it does not express exactly situation, angles and movement.... But this new characteristic....cannot fail to give at the same time the solution, the construction and the geometrical proof, the whole in a natural manner and by analysis. That is by determined ways."

stand any of his theorems. "As in the analysis demonstrated here every equation is only the expression, clothed in the form of the analysis, of a geometrical relation, and this relation expresses itself clearly in the equation without being obscured by arbitrary magnitudes—as for example the coordinates of the usual analysis—and therefore can be read off from it without further trouble; and as further every form of such equation is only the expression of a corresponding construction, then it follows that as a matter of fact, by means of the analysis here given, the solution of a geometrical problem results at the same time as the construction and the proof. As further nothing arbitrary ....need be introduced, the kind of solution must always be according to the nature of the problem; and as it is in the form of analysis, therefore a necessary one in which there can be no question of any seeking round for methods of solution." In other words the fusion of synthetic and analytic methods which Leibniz hoped for is fully accomplished.

It must be noted that in one respect Grassmann has not only realized the prophetic vision of Leibniz but also cleared away the inconsistency which vitiates his attempt at making his dream come true. For Leibniz, seeing that the *fundamental* analysis of geometry must rest on non-metrical relations, yet desired its *final* application to mechanics and natural science, in which metrical relations are all important. So he was led to a half-hearted attempt at non-metrical analysis by means of a relation—congruence—which, while showing the way to a geometry not based on algebra, yet failed itself to travel far in that direction. The special merit of Grassmann has been to found a geometrical analysis free of magnitude and yet so to develop it that metrical considerations may be introduced without disturbing the form of that analysis. Projective geometry, therefore, only partly fulfils Leibniz's hopes;

their complete realization is found in Grassmann's theory of extension.

§ 11.

We began our discussion of the relation between Grassmann's calculus and the characteristic of Leibniz by an analysis of the manner in which their work has been received by the average mathematician. It seems to me that we can profitably return to these historical considerations for a moment, and look at them from another view-point.

There is some reason, as I have tried to show elsewhere,[28] for citing lack of historical perspective on the part of mathematicians as the cause of the unsympathetic attitude commonly taken up in regard to work of philosophical breadth; and that if more regard were paid to historical development in mathematical education wider and more penetrating vision would result. The position taken up is well expressed by Branford[29]: "The path of most effective development of knowledge and power in the individual coincides, in broad outline, with the path historically traversed by the race in developing that particular kind of knowledge and power." At the same time, however, we must realize that, if we alter our attitude to this slightly, and regard it not from the point of view of the educationalist but from that of the original worker himself, obsession with origins seems inevitably to lead to what I have called above the genetic error. The effective point of departure in attaining knowledge of geometry may be from such empirical and utilitarian experiments as form its historical origin. But that must not be allowed to create an atmosphere hostile to any recognition of the *a priori* and formal nature of that science.

Furthermore the historical method may lead to a certain *ex cathedra* manner, a reliance on authority and tra-

[28] "The Neglect of the Work of H. Grassmann."

[29] B. Branford, *A Study of Mathematical Education*, 1908.

dition. It is this factor which especially concerns us in our attempt to see the work of Leibniz and Grassmann in true relation to each other and to mathematical thought. For Couturat points out[30] that what probably hindered Leibniz's development of his geometrical calculus and rendered abortive his attempt at its realization was the authority of Euclid. He says, "Why, amidst all the relations which Leibniz catalogued, did he give preference to the relation of congruence and neglect the relations of similarity, inclusion, situation, which serve to-day as the bases of quite new sciences[31] which he foresaw and would have been able to found? It is evidently because tradition, represented and embodied by the *Elements* of Euclid, limited geometry to the study of the metrical properties of space. Now the tradition is not explicable by any reason of *theoretical* order (considering that metrical relations are more complex and less general than projective relations) but solely by reason of historical and practical order."

I have already in the previous section shown that another explanation may be held of this clinging to a metrical relation by Leibniz. However that may be, the authority of the Euclidean tradition may have had some influence on his work in geometry, as the Aristotelian tradition had in his foreshadowing of a logical characteristic.[32] In fact we shall not be laying over-emphasis on the tendencies of an exaggerated reliance on historical method if we say that its final result is the attitude of the young critic in Shaw's play[33] who says, in effect, "Give me the name of the author and I'll tell you if it's a good play." If that critic held a university chair of historical criticism he would doubtless be able to find valid arguments for his position—for how

[30] *La logique de Leibniz,* pp. 438-440. Russell however attributes Leibniz's failure to his holding the relational theory of space, *Mind,* 1903, p. 190.

[31] Theory of aggregates, modern *Analysis situs,* projective geometry, etc.

[32] See note 7 above.

[33] "Fanny's First Play."

(he might ask) can one judge competently without a complete set of data, and is not authorship an important datum? It is irritation at this standpoint which causes Mr. Bertrand Russell, whom I have heard speak very forcibly on the subject, inveigh against this hyper-historical method. But the objection can be stated in a much stronger form. "Erudition often does violence to inventive power: and the proof is that the modern discoverers of symbolic logic, Boole and his successors, have all ignored (and rightly) the example and precedent of Leibniz; it has even been remarked[34] that they have almost all been ignorant of one another, and if this ignorance has been a source of error, it has been above all a condition of originality."[35] Now it does not appear to me that the essential defect of such an extreme anti-historical attitude has been that it caused error. Staudt realized the ambitions of Leibniz in some degree in founding his projective geometry, and Grassmann in still further degree in creating his theory of extension, without knowing that their historical origins lay in his work. No great harm comes from this, although an original genius will, as a general rule, be less likely to be deflected from his way by the work of others than to find in them sources of stimulation. But to the mass of us, who form the bulk of mankind, narrowness is a mental blinker which hides the full splendor of the creations of genius. The real toll taken by historical ignorance is in neglect of originality, and the loss of power and influence consequent on it.[36]

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[34] J. Venn, *Symbolic Logic,* Introd., pp.29, 30.

[35] Couturat, *op. cit.*, p. 440.

[36] I have to thank Miss Vinvela Cummin for valuable help in translating the *Geometrische Analyse.*

GREEK IDEAS OF AN AFTERWORLD.

A STUDY OF THE RELATION BETWEEN PRACTICE AND BELIEF.

WE have a free and easy way of generalizing the afterworld of Greek religious belief as an underworld. This is indeed the usual form of the belief from Hesiod onward, and it is the view generally disclosed by Homer both in the Iliad and Odyssey. Yet the fact is that the most deliberate and detailed Greek presentation of the approach to that dread world, that of the eleventh book of the Odyssey, does not at all represent it as an underworld like the infernal regions of Vergil's fancy, but as a far western realm. The far-wandering Odysseus sails to the distant west, out of the sea, and across the mighty ocean stream to its farther shore; he beaches his black-hulled ship on a lone waste beach where stand the barren groves of Persephone; thence he directs his steps inland to a great white rock at the confluence of the Styx, Pyriphlegethon and Acheron; and there it is that he enters the purlieus of the many-peopled house of dark-browed Hades.

The Odyssean realm of the dead is reached neither by descent into a cave nor by passage underneath an overhanging ledge. It is of the same level as the land of living men. Its darkness is apparently due to its location beyond the path of the western sun, which, descending into Ocean Stream, disappears somewhere from the sight of mortal men to be ushered in anew by rosy-fingered Eos, each succeeding morn. To speak of Odysseus as descending into an underworld is to have but little regard for the language of Homer. Clearly to discern the picture that he actually

presents is to become aware of a striking contrast between it and the afterworld of classic Greek and Roman belief; and this contrast raises the problem of explaining and accounting for such different views, obviously related in the same way to the same fact,—the fact of death. An obvious relation, I say; if this appears to be but a bold assumption, I trust it will be justified in the course of my argument.

A study of early man's beliefs about an afterworld involves a consideration of two groups or series of facts—mental facts and motor facts, or facts of belief and facts of practice,—both associated with the event of death. Apparently these two kinds of facts do not simply constitute two parallel series that were mutually unrelated in life and thought and that may therefore be studied and understood apart from each other; they seem to have an intimate and genetic relationship. This, however, is not to say that they are absolutely simultaneous in origin, or that one may not be primary and the other secondary, both in origin and importance. On the contrary, in their genesis, either belief is antecedent and causal to practice, or practice is antecedent and causal to belief.

It is popularly supposed that belief originates and dictates practice, or custom, which is thus regarded as secondary to belief. Anthropologists generally confirm the supposition, and whole systems of social interpretation and philosophy have been built upon the assumption. Professor Seymour, in his *Greek Life in the Homeric Age,* insists upon this relation in the case of Greek mortuary practice and belief, and cautions the reader against assuming that the Greeks who maintained certain customs may have "inherited also the beliefs on which those customs were originally based." He brings to bear upon the case the authority of the German scholar Rohde, declaring that "Rohde gives as the *cause* of the adoption of cremation by the ancestors of the Homeric Greeks, a desire to rid themselves

of the souls of the dead; and as a *result* of the change, the abandonment of the old ritual and sacrifices."

According to Professor Brinton, "The funeral or mortuary ceremonies, which are often so elaborate and so punctiliously performed in savage tribes, have a twofold purpose. They are equally for the benefit of the individual and for that of the community. If they are neglected or inadequately conducted, the restless spirit of the departed cannot reach the realm of joyous peace, and therefore returns to lurk about his former home and to plague the survivors for their carelessness.

"It was therefore to lay the ghost, to avoid the anger of the disembodied spirit, that the living instituted and performed the burial ceremonies; while it became to the interest of the individual to provide for it that those rites should be carried out which would conduct his own soul to the abode of the blest."

Here again practice is regarded as secondary to belief, and is interpreted by reference to belief. Professor Frazer, also, the dean of living anthropologists, insists upon this relation between our two series of facts, and cannot admit or conceive of the opposite as being true. I intend, however, to take the other side of the question here involved, advancing the proposition that it was mortuary practice that constituted the motive for belief in an afterworld; and especially shall I endeavor to indicate the application of this formula to the genetic interpretation of Greek ideas of an afterworld.

The Hellenic peoples of whom we have knowledge universally believed in an afterworld, whither the souls of mortals departed at death and where they had a continued existence. But they entertained not merely the two conflicting beliefs already mentioned; they held in developed form at least four quite different beliefs regarding the destination and abode of the souls of the dead. According

to one of these beliefs the souls of dead men ascended to Olympus, as did that of Heracles in story; according to another they descended into an underworld; in the eleventh book of the Odyssey Homer places them in a continental region beyond the western verge of Ocean Stream; and Pindar places the souls of great heroes in "Islands of the Blest" in the far Western Ocean.

It may be well at this point to note some apparently fundamental resemblances between these last two beliefs. Pindar places the souls of sinful mortals in an underworld, subject to sentences reluctantly imposed upon them. Hesiod declares that the men of the Golden, Silver and Bronze ages were hidden away in earth; and it is but natural, because of the different types of life imputed to them, that he should fancy different conditions for them after death. But the souls of his age of heroes, he says, were given a life and an abode apart from men, and established at the ends of the earth in "Islands of the Blest by deep-eddying Ocean." He does not state the direction of these wondrous islands, but undoubtedly their direction, like that of the Pindaric Islands of the Blest and that of the Odyssean realm, was already so fixed in the tradition of his day that there was no need of indicating it. It would appear, then, that in essential characteristics the continental Odyssean realm and the Islands of the Blest are alike in being conceived as western, and differ only in geographical form and extent. From this it would further appear that the notions of these two similar abodes of the dead are variants derived from a single source. But if these two notions did grow out of a single origin there was certainly a reason for the divergence, which it should be part of our task to discover. And yet, on the other hand, it may be unnecessary or even incorrect to assign their origin to the same people, even though we may feel compelled to assume

that the significant common element of direction must inhere in a common element of antecedent cause.

Whence came these three or four differing beliefs? That is to say, upon what difference of psychological ground do they severally stand? No one man could at one time entertain so many and so contradictory beliefs upon one subject; neither could one homogeneous people, as, for example, a single city state of the Mycenean civilization, or even the Minoan civilization of Crete as a whole. Wherefore we should probably look for this difference of belief either in the several racial stocks amalgamated to form the historic Greek people, or in part to their respective traditional beliefs and in part to alien streams of influence. But in either case it will be pertinent to inquire how different races and racial stocks should have come thus to act and believe so differently in the face of the same fact, death. To trace a belief or practice from one people back to another should never be taken as an explanation; this done, the question of real origin and motive still remains, as insistent as ever. Neither should identity of belief or practice be taken as necessary evidence of racial relationships, or of racial contacts; nor difference of belief or practice as evidence of difference of race. There are others besides the human factor that enter into the origin and development of practice, as we shall presently see.

With regard to mortuary practice, the Greek world furnishes only two types of historically attested facts. The Homeric Achæans cremated their dead, and the practice survived far beyond the Homeric, and even the Periclean age. The Mycenean civilization laid its dead beneath the surface of the earth, and this practice gradually superseded cremation, even among the descendants of the Achæans. Thus the Greeks of historical times had two strongly contrasted modes of disposing of their dead, corresponding to two of the contrasted beliefs we have mentioned. For there

is undoubtedly a genetic relation between cremation and belief in a heavenly abode of souls, and between inhumation and belief in an underworld. But which is cause and which effect? And how did the causal series itself originate? And how could the belief in a western abode of souls be related to either of these, either as antecedent or as consequent?

These two series of facts in Hellenic life give rise to three problems of immediate significance; to say nothing of others more remote, as for example, how man came to believe that he had a soul at all, how nearly the belief coincides with actuality, the origin of religious fears, etc. The three special problems thus isolated for present consideration are:

1. What is the genetic relation and order of precedence between practice and belief,—between cremation and belief in a heavenly abode of souls, and between inhumation and belief in an underworld?

2. In case either belief or practice is found to be antecedent to the other, how then did this antecedent series take its rise?

3. Whence and how came the belief in a far western abode of souls, and why the apparently twofold differentiation of this belief, which we have noted?

In the interest of brevity I may appear to be cutting the Gordian knot rather than untying it; but I feel sure that the drift of my argument will be caught, and that its essential truth must make a strong appeal for assent.

In the first place, let us consider this intimate and inherent correspondence between mortuary practice and belief about the dead, under conditions where we can see more plainly the part played by geographical environment, and where at the same time we can be sure of the soil on which our two series of facts originated; for we know not yet where the practice of inhumation originated among the

Myceneans and Minoans, nor where cremation first developed among the Achæans.

The ancient Egyptians and the Incas of Peru preserved their dead by mummification, and both believed in a bodily resurrection of the dead. We are reasonably sure that the land where each of these peoples developed was likewise the soil upon which their respective traditions in this matter originated; we shall be still more sure of this local origin as we proceed. Did the belief or the practice precede?

Now no matter what we may imagine them to have thought about soul and body and their mutual relations before the practice began, the Egyptians and Peruvians could not have cremated their dead; both Egypt and Peru lacked that abundant supply of fuel which would be necessary for this practice among a numerous people. Neither could either people long have inhumed its dead in the fertile valley land of its abode. These restricted valleys early became the seat of such dense populations that productive land could not be permanently set aside for burial purposes; nor could land under cultivation be wantonly trampled over for this common social purpose, even though six feet of earth were sufficient for the individual grave. Of necessity, therefore, the adjacent desert ridges were employed for the purpose, and the earliest mode of burial there was inhumation. But the dry climate and the nitrous character of the upland soil, both in Egypt and Peru, tended naturally to preserve the bodies of the dead. The action of wind and wild animals, however, tended often to exhume them, at the same time disclosing a high degree of preservation. In order to protect their dead, especially to prevent the work of their hands from being made of none effect, the Egyptians, in particular, came to build rock tombs. But this required much labor and expense. Yet it was cheaper to build one tomb large enough for many burials,

for whole families, even through successive generations, than to build many individual tombs. Hence, by mutual suggestion and social rivalry through long stretches of time, the mighty Pyramids of Egypt came to be developed.

But under these conditions a tomb must be entered from time to time for new burials; and in spite of their high degree of preservation by natural means, the bodies of the dead within gave rise to noisome odors. Hence arose the practice of embalming with aromatic spices, to counteract or obscure the evil odors of decomposition. What but this fact of unpleasant odors could first have suggested the use of expensive spices in embalming? With the prominent Egyptian nose was undoubtedly associated a keen sense of smell. Wrappings of linen served in the first instance to retain the spices. The embalming tended to more perfect preservation of the flesh, and this result also helped to accomplish the primary object of the practice, which was the laying of unpleasant odors. Upon this combination of facts arose a profession of embalmers, who developed a more and more elaborate technique. When death and funerals had thus become an economic burden upon the living, for which no obvious or adequate return was received, the question of meaning inevitably arose and persistently pressed for a satisfactory answer. It is exceedingly difficult for man to admit that he is spending sacred energies in vain or purposeless quests, and thus making a fool of himself; and so the practice, entailing so large an expense, insistently required a sanction, and a tremendous one at that.

Now the care lavished upon the dead body, by tending to preserve it for an indefinite length of time, embodied within it an inherent and obvious suggestion of the primary sanction that actually came to be formulated. For by this time embalming had come to take place before the process of decomposition had set in; and the original cause of the practice was no longer making its appearance, even though

from allied experience the agents may well have been aware of what would soon happen without embalming and burial. So now, instead of really knowing that they are trying to forestall or allay the noisome odors of decomposition, they detect but one purpose in the practice, the preservation of the body. But why should the body of the dead be preserved? With this query arose the first suggestion of a mystical or transcendental idea in association with the practice, and the first attempt to formulate an ultra-pragmatic or other-world sanction for it. This sanction was formulated as an explanation. It was from the first employed for this purpose, and as all thinking individuals were implicated in the practice no one was in a position to question or challenge it.

It might be urged on this latter ground that the question of purpose or value could never have arisen; but we must not overlook the fact of foreign contacts—especially among the Egyptians—wherein contrasted practices would raise the question from without, if not from within. Besides this, they had always the poor with them, who, from contrast with their own meager efforts in the same field, would be forced to think about values. And above all, there was always growing up among them the supreme pragmatist,—the eager, curious child.

Thus this question of values, like the ghost of Banquo, was ever likely to confront the living, and only a powerful sanction would serve to lay it. The priesthood and the professional embalmers, in particular, had constant need of the sanction, as a means of justifying their existence. Thus it is that this sanction arose, and that it has been passed on and received as an explanation even by the wisest, even unto the present day. And that is in brief the story of the Egyptian and Peruvian practice of mummification, and of their belief in a bodily resurrection. It all comes back in the last analysis to the fact of decomposition

and the despised sense of smell, which would move men to acts of aversion and riddance.

But, one may ask, is it not after all just possible that this practice arose out of an antecedent idea of souls and the notion that the body must be preserved against a future resurrection and a reincarnation of the soul? Rather is it not far more reasonable to see that the belief arose out of the practice, as a sanction for the care and expense involved in it? On the first alternative we must certainly congratulate the Egyptians, and the Peruvians too, on having found a geographical location so congenial to their belief. What would they have practiced, or how could this belief have survived, had they lived in the valley of the Congo or Amazon, or even in Greece? Or how could they have come to believe in a heavenly abode of souls, when they did not cremate? And if the belief in a bodily resurrection came before the practice of mummification, then how did this notion and belief arise?

Now let us take a look at barren, hungry, frost-bitten Tibet. What burial practices and what cognate beliefs about the dead have from the first been inherent in the natural environment of man presented by the Himalayan highland? Let us picture to ourselves a people making here its arduous ascent from lowest savagery to barbarism. As they come to have a settled place of abode, how shall they secure for themselves riddance from the discomforting odors of decomposition that follow in the train of death? Suppose that they have attained to such a degree of economic efficiency as to have left behind the practice of cannibalism, and that they are as yet without any metaphysical or transcendental ideas and beliefs; how then shall they dispose of their dead? Or what shall they believe about their dead, if they have as yet paid no attention to them save by the simplest modes of seposition and abandonment?

Here in Tibet is a people that could not cremate its dead; for here, too, fuel is scarce. Neither could it inhume its dead; for during a considerable portion of the year the deeply frozen ground is proof against even the tools of civilized man. Here preservation of the dead by natural means, that of freezing, may be assured for a season; but should this be relied upon temporarily, final burial by one means or another would become imperative with the advent of spring. Shall the Tibetans preserve the bodies of their dead through the long winter, to the end that they may give them some sort of approved burial in the spring? What could originally have suggested to them the notion of an approved form of burial, and of the preservation of their dead against the time when this should become possible? The primary function of burial by whatever means is avoidance or riddance of certain after effects of death; and with an abundance of carnivorous animal life scouring the country for the means of subsistence, how could the immediate, practical function of burial be more readily or more easily secured than by calling in the aid of dogs and vultures that infest the land? Now this is exactly what the Tibetans do, even to-day. And from their own hard struggle for existence they furthermore feel it an act of charity thus to minister to these scavengers of their land. There is no other people on earth with whom charity is so highly esteemed as a virtue, and so universally encouraged. Under the hard conditions of life, charity, generosity, is a necessary practice among their own kind. And furthermore, the leisure-class priesthood, which is very numerous, in its own self-interest has need of encouraging this fundamentally necessary virtue; and finally, this virtue is invoked as a sanction for the feeding of their dead to the beasts of the field and the birds of the air. Without some notion of other ways of securing this same object, they could feel no need of this or of any other sanction.

In the case of a very few individuals of the highest rank cremation is allowed as a special honor, and naturally this privilege is mostly restricted to the religious hierarchy. It is evidently not the native Tibetan practice, but was plainly introduced into Tibet by the Buddhists of India, with whom it was native. But the great majority of the Tibetan dead go to feed the hungry dogs and vultures, which are highly esteemed for this purpose; and this, despite the fact that Tibet has for a dozen centuries been subjected to Buddhist influence, which would naturally favor cremation, its own native mode of burial, if this were economically possible. Here in Tibet the native mode of burial is directly apposite to geographical conditions, even as it was in Egypt and Peru; and the beliefs by which it is explained are merely so many sanctions, or justifications, which have been developed out of the practice itself.

But what are the Tibetan beliefs about the dead? When once they have acquired the notion of a soul that survives the event of death, whether originally or by adoption from other peoples, we should expect them to hold a belief in some sort of transmigration. From seeing the bodies of the dead devoured by animals, they would seem naturally to think that souls also passed into the bodies of these living sepulchers. This is exactly what they believe. We should furthermore expect them to have a preference for transmigration into the winged vulture that sails so easily through the air, to taking up their abode within the body of a lazy, grunting pig or snarling dog. Here too our surmises are correct. In the course of centuries, as the relation between practice and belief has become obscured, their beliefs have been elaborated and graduated, so that even non-carnivorous animals are included in certain cycles of transmigration. But in this fact of feeding their dead to animals is certainly to be found the original germ and suggestion of their belief in transmigration. Tibetan religious

ideas and beliefs are not so definitely conceived nor so systematically organized as are those of some other peoples, because their authors have never devoted so much personal care and energy to the disposal of their dead. They have not felt so strong a necessity for justifying their practice as have the Egyptians and some other peoples.

Again, let us consider the case of India, where Brahmanism and Buddhism have their origin and home. The Indians, like the Tibetans, hold a belief in transmigration, and of course for that same fundamental reason. That a mighty, far-scattered people like the Indians exhibits a characteristic belief or practice does not mean that all individuals of the group hold it in common. It would be too much to expect such a people, or any people at all, to be really homogeneous in belief and practice from the early stage when human burial began among their forebears until the present time. Thousands of families in India today are too poor to afford the most characteristic traditional form of burial for their dead, and throw them into rivers, or otherwise dispose of them. In Indo-China those to poor to afford cremation commonly carry out their dead to be eaten by the beasts of the jungle. On the Ganges, "When the pyre is built the nearest relative of the deceased goes to the temple and haggles with the keeper of the sacred fire over the price of a spark; and having paid what is required he brings the fire down in smouldering straw and lights the pile. If the family can afford to buy enough wood, the body is completely consumed; in any case the ashes or whatever is left on the exhaustion of the fire is thrown into the sacred river;. . . .and any failure on the part of the fire to do its full duty is made good by the fish and the crocodiles."[1] Thus it is easy to see how in bygone days the Indian, at least in the lower social strata, became possessed of a belief in transmigration, and how, through

[1] Pratt, *India and its Faiths.* New York, 1915, p. 44.

ignorance of its primary source and relationships, carried it over into relationships bearing little or no connection with its parent practice, as in his abstinence from eating flesh.

And yet India, with its wide extent and countless population, has more constant elements of religious and philosophical belief than would at first seem possible,—a result of mutual contacts and social cooperation through long stretches of time. "The central point of Hindu thought is the soul. It is from the soul or self that all the reasoning of the Hindu starts and to it that all his arguments finally return."[2] Probably the most widely known characteristic of Indian religious philosophy is the doctrine of the immanence and absoluteness of the supreme soul Brahman, with its correlate doctrine of the oneness of the individual self with the All,—the merging of the objective, phenomenal world into the universal absolute, which is Brahman. Yet it is plain that this interest in the objective world begins with the individual human self. "This unity of the soul with God is at the foundation not only of Hindu metaphysics, but of Hindu ethics as well. The great aim of life is the full realization of that God-consciousness, the significance of which forms the central point of Hindu thought. Before this can be fully attained, the soul must be liberated from the mass of particular interest and petty wishes and self-born illusions which weigh it down and hide from it the beatific vision. Hence *liberation* and *realization* may be called the twin ideals of Hinduism, and it is these that determine all its ethical theory."[3]

The doctrine of "liberation" and "realization," the doctrine of Nirvana, the yoga-systems, and other characteristic Indian notions would be meaningless and impossible without the basic body of "religious intuitions" that make

[2] Pratt, p. 91.

[3] Pratt, p. 92.

up the Brahmanistic doctrine of the Upanishads. But an "intuition" has in primordial genesis some sensuous basis, direct or indirect; and so, instead of seeking for the idea or philosophy back of the practices associated with these and other beliefs, we should undoubtedly seek in practice for the sensuous elements of suggestion that formed the basis of the beliefs, and then seek in turn for the sensuous motive of the practice itself.

We may admit that the Indians are a peculiar people; yet, when we pin ourselves down to minute details, we note that the testimony of their senses, the ultimate constituent of all intellectual forms, is the same as our own. Their intellectual peculiarity consists not in their physical or psychological selves, but in the differences of their objective environment, part of which they themselves make, and in the various ways in which the sensuous details of experience with it have been combined through generations of spontaneous social collaboration.

If, then, we consider these doctrines of the "infinite ocean of the absolute Brahman"; of the essential oneness of the one with the All; of the soul's struggle for liberation to realize and complete this oneness in "Nirvana, or reabsorption into the eternal light": as we contemplate these doctrines, seeking to discover their source in sensuous experience at a time antedating the rise of science with its theories of atoms and corpuscles, can we not almost see before our eyes the primitive populace of India cremating its dead and beholding the body ascending in the form of flame and smoke, thus becoming absorbed in the ocean of air, which to them, at that time, seems infinite?

We examine Indian burial practices, both present and past, and we find that from time immemorial cremation has been a characteristic Indian mode of burial. When men actually beheld the body of a deceased friend dissolve and mingle with the elements, they were bound to have

different thoughts about the destiny of the individual than if it were laid away in earth to decompose by degrees for an unknown length of time, or if it were altogether preserved by embalming against decomposition. And from seeing the individual thus pass so visibly from a corporate existence into thin air, they would also be moved more strongly to contemplate the other end of individual existence, the whence as well as the whither. There could be no doubt that the deceased had attained to freedom from the bonds and ills of terrestrial existence; and the living, from their own desires to live beyond the usual limits of life, would be brought face to face with the question whether they should ever live again and how their scattered selves could realize another conscious existence. To hold before them the notion of another life as something to be desired was to believe in it; and from this point it was an easy matter to identify the conditions of existence before birth and after death, whence Brahman becomes the source, the end, and the essential constituent of individual existence.

Add to all this the practice of feeding to animals either the entire body or the remains of partial cremation, already noted—the differences of practice being characteristically in agreement with differences of social rank—and we have the proper sensuous background in practice for the doctrine of transmigration, which we find embodied in the doctrine of Karma and fused with the doctrine of Nirvana.

Geographical conditions undoubtedly favored cremation in India in the days when fuel was abundant and easily secured. But with a numerous population making large demands upon the wood-supply through scores of generations, the practice has become more and more expensive, and the demand for sufficient sanction has become more imperative. Thus in the course of many centuries the beliefs genetically inhering in these practices have become much elaborated; and, by the development of an elab-

orate logic and metaphysic, they have in turn modified the practice itself. It is in this way that the religious institution has justified its ways and made itself indispensable to men.

Here again we may claim without fear of successful contradiction that burial practice arose as a purely practical matter and by its form dictated the form that belief about souls must take, when once the notion of soul itself arose out of the practice. The sense of smell together with the simple, practical knowledge of the purifying agency of fire suggested and motivated the practice; here it is that we find the sensuous motive behind the practice, which in turn motivated the belief. Primarily, the belief is a supposed explanation of the practice, invented when the practice had become so highly elaborated as to conceal its real cause and thus to demand justification. Men do not feel the need of explaining or justifying the obviously practical.

But the explanation given of this and other kinds of practice is not an explanatiaon of the covert act; rather is it intended to explain or justify the care and energy devoted to it or required by it in the name of social form. The overt act merely affords suggestions toward the explanation that is evolved. It is only after a long lapse of time during which a practice has by social concurrence become highly elaborated that a justification is required. Men acting in unison, with a common sense or emotional interest, will do extravagant things not dreamed of in individual life. But, having participated in such an act, unsophisticated man can easily find a justification for his act, suggested by the act itself. It seems to be a characteristic of universal human nature, in the absence of a true, antecedent cause for specified conduct, to seek about for some consequent justification; and the race seems equally prone

to accept such a justification as a statement of antecedent cause.

And now we may return to the case of Greece. We do not find there that close, almost necessary relation between practice and environment which we have seen in Tibet, Egypt and Peru; in fact, we cannot say with certainty where the two historic Greek forms of burial originated. Already some 3000 years before the Christian era we find the Minoan civilization in the Ægean world, practicing inhumation. And the northern Achæans, from whatever source they came, were already at their arrival in Greece practicing cremation. As to the relation between the beliefs and practices that prevailed on Hellenic soil, we can argue only by analogy, or homology, with what we have seen to be true in Egypt, Peru, Tibet and India; but it is far more reasonable to believe that the same relation holds true here than to defend the other horn of the dilemma.

With regard to the Achæan belief in a heavenly abode of souls, we may cut the matter short by asserting its rise out of the practice of cremation. In the course of time, after this practice had become the rule among the ancestors of the Homeric Achæans, they probably came to feel much the same regarding it as did the Indian of California. "It is the one passion of his superstition to think of the soul of his departed friend as set free, and purified by the flames; not bound to the mouldering body, but borne up on the soft clouds of smoke toward the beautiful sun."[4] I say the Achæan may have come to feel in this way, much as did the Hindu; but this was not the original motive of his practice. His thoughts about the mouldering body of his departed friend and his fancies about purification were not in the first instance inspired by a desire for the friend's welfare after death; he was first of all concerned for the

[4] Powers, *The Indians of California*, pp. 181, 207.

living, especially with regard to the sense of smell. And however transcendental the notion of purification came to be by reinterpretation of the practice, after its original motive had ceased to prevail—because burial came to be practiced before decomposition had set in—the very association of purity with cremation betrays the original motive of the practice, just as did the use of spices by the Egyptians.

As with cremation among the Achæans, so in the case of inhumation among the Minoans and Mycenæans we may assert that the practice was suggested, and passed through its primary stage of development, as a means of escape from the discomforting odors of decomposition. And as the belief in an upper-world abode of souls developed as an explanation and sanction for cremation, so belief in an underworld developed by suggestion from the practice of inhumation. To make good the claim that belief came first and suggested practice, one must show satisfactorily how any people ever could have associated souls with a heavenly or with an underworld abode without the practice of cremation or inhumation, respectively, or at least contact with some people who did practice this mode of burial.

The belief associated with cremation never became so highly elaborated in Greece as it did in India, and for very good reasons. For in the first place, Greece never came so completely into the power of a priestly class as did India; and in the second place, the practice on which it depended here came into rivalry with the already established practice of inhumation, which on the whole was cheaper. To this we should add the fact that the social institutions of the older race proved to be the more persistent, as with the Normans and Saxons in England, whence this must have been especially true of such ideas as we are discussing. And however spectacular and interesting the act of crema-

tion became among the Hellenes, as reflected in the Homeric picture of the funerals of Patroclus and Hector, the accompanying conception of the soul after death could be but very vaguely imaged, as in the case of India; while the same idea accompanying burial in the ground, in cave-tombs, cist-tombs, and rock-tombs, as the so-called "treasury of Atreus" was capable of very definite imagery. Thus, although cremation continued to be practiced side by side with inhumation, it was the belief associated with the latter practice that possessed the more definite imaginative appeal, and that finally prevailed.

Yet the upper-world conception of the soul persisted and influenced the belief of later generations. As in the first instance it was only the Achæan masters of Hellas who practiced cremation, while the subject populace inhumed its dead; and since in the classical age it was only the wealthy who could afford cremation; so it came to be believed that the "good"—the worthy and the proud—at death went to heaven above, while the poor in purse and spirit descended into hell. Various modifications of this composite belief have grown up by internal suggestion and by accretions from foreign practices and beliefs; but in the last analysis each belief grew out of a practice, and the practice originated as an obvious and immediately practical necessity.

While we cannot say just where or why the Minoans developed inhumation and the Achæans cremation, or why some other practice did not arise and prevail among each people, yet it is perhaps significant that cremation was the practice of the northern race, like the aboriginal Hindus, —a people who had more need of fire on a large scale, such as would be necessary for the cremation of human bodies, a people with whom fire was necessarily a more continuous object of experience and therefore a more constant agent

of purification in other ways also, than it was in the sunny southland of Crete and Hellas.

Homer was the poet of the Achæan overlords of Hellas. Yet he was apparently not of the Achæan race. Although he quite consistently presents to us the Achæan mode of burial, his idea of the soul and its abode is not consistent with the practice of cremation. He thinks of the cremated Heracles as having a corporate existence in Olympus, with lovely-ankled Hebe at his side; yet Heracles must also be seen of Odysseus in the house of Hades. Homer is himself aware of the contradiction, and declares it to be but a phantom that Odysseus sees there. On the other hand, Achæan heroes—Patroclus and others such as would naturally have been cremated—he unequivocally represents as being in the populous realm of Hades in the distant west. In Homer's references to the realm of the dead we discern the unconscious and inextricable mingling of at least three traditional views on the subject. Nor should we be surprised at this when we note that the entire period from the Trojan War to the final completion of the Homeric tales was one of ethnic amalgamation between at least the two races we have already mentioned. Our view of this process is still further complicated, and yet perhaps much illuminated, by the knowledge of a continuous intercourse with the west coast of Asia Minor during this time, such that most of the cities that laid claim to Homer were of this region.

And this prompts us to consider how the notion could have arisen that the dread abode of souls was in the west. It would perhaps be interesting to point to the west as the region of the setting sun, to associate it with the death of the day, and to conjure up some fancied analogy as having been indulged in by the aboriginal authors of this tradition. Yet in the face of such a procedure stands the fact that the west has always been the land of allurement and promise

to which Greek no less than Teuton has ever turned his eyes. The fact is that if the association of the west is an essential element of the belief, as it appears to be, then thoughts of the west were inherently involved in the form of burial with which the belief was genetically associated.

We might look to cremation for the source of the association, if anywhere in the Ægean world the prevailing winds blew to the westward, thus bearing the smoke of the funeral pyre in that direction. But such is not the case; and besides, neither the earthly location of the Odyssean afterworld and the Islands of the Blest, nor the substantial, corporeal nature of the spirits dwelling there would permit of this conclusion.

I know not what may be the value of the suggestion I am about to make upon this subject; I simply present it as the most plausible explanation I can imagine for the conception of a western realm of the dead. I have by no means enumerated all the methods that man has employed for the disposal of his dead. Fundamentally there is but one reason for disposing of the dead by any means, and that is to secure a separation between the dead and the living. Inhumation and cremation are merely the most obvious and most universally practicable means of securing this one end.

Now one of the simplest modes of accomplishing this object, where natural facilities permit, is what is called canoe-burial,—a mode in which the body of the dead is placed upon a log, or raft, or boat, and set adrift upon the sea, or down a stream. In the course of time this practice, just as any other, is subject to elaboration and refinement, and finally to mythical, transcendental interpretation. I suggest that this Hellenic notion of a western realm of the dead originated on the western coast of Asia Minor. Here all rivers flow to the west; out to the westward over the sea are beautiful islands which could once have been imag-

ined as the destination of bodies set adrift on the rivers of this coast; and finally, when these islands had been visited and explored and the fancy exploded, it was but natural to set the place of destination of the dead still farther to the west beyond the Ægean archipelago. And since even by Homer's time the Hellenes had dim fancies, more or less substantiated, of extensive coasts in the distant west, it was but natural that the earlier notion of an island abode for the dead had to give way to fancies of a more continental region. But as the primitive occupants of this Asiatic coast had grown bolder and put out to sea, they had perhaps found on the coasts of the Ægean islands the unsightly wrecks of their death-craft, and so had come to discontinue the practice. It is not necessary to suppose that this practice was current in the time of Homer, or even of the Trojan War; mythical fancies may survive long after the conditions that fathered them have ceased to exist.

Such is my suggestion for explaining the notion of a western abode of souls, presented on the assumption that both these traditions go back to a single local source. Yet I am not unmindful that the coast of Epirus and Illyria furnish the natural conditions in which either one or both may have arisen; whence we should have to suppose that they were brought into Greece by the Achæans. On this assumption we should have to suppose further that these Achæan adventurers, after leaving their native abode and the conditions supporting their native mortuary practice, took to cremation as a new means of disposing of their dead, and yet retained the tradition associated with the native practice of canoe burial. This would help to account for the incongruities in the Homeric conception of the condition of souls whose bodies had been burned; it would mean that they had not yet maintained the practice long enough to have invested it with a systematic sanction and

philosophy. As between these two suggestions, I should probably prefer the former. As yet I see no way in which archeology may help us here.

In any case the tradition of a western abode of the dead, which had already been started and which had by this time lost all direct association with the practice, continued and gathered to itself the Homeric, and Hesiodic, and Pindaric refinements and differentiæ which we have already noted. Such is the regular course of tradition. It is undoubtedly in this way, and by reference to the same kind of burial practice in Britain that the traditional picture came to be built up of the black-hulled ship that bore "Elaine the fair, Elaine the beautiful" down the Thames to Westminster; and of that other dusky barge that bore out into the mystic lake beyond the ken of mortal man all that was mortal of good King Arthur. Such a social background is probably necessary for the historical interpretation of the death voyage of Sinfiotli, son of Sigmund, away "to the west"; and of Balder and his faithful wife Nanna, laid on their funeral pyre on the deck of the stately ship Ringhorn. We can understand and explain how a traditional practice arises and grows by social concurrence, and how a belief arises in association with it, all conscious association with the practice being gradually lost. But to explain how practice should arise out of an antecedent belief, and how that belief should first have arisen as a purely intellectual conception without sensuous motivation—as the grin without the cat, as one might say—in spite of some three thousand years of effort upon this problem, we are quite as far from a satisfactory solution as ever.

To conclude, then, the act of burial by early peoples is an act of aversion and riddance, even as the traditional interpreters of the act have claimed; but the primary object of the riddance, instead of being a metaphysical, or spiritual object, is a real, concrete, sensuous reality, which is

exactly the necessary and apposite kind of motive that we should expect. If only Hobbes had hit upon this formula! But he had not at hand the rich accumulation of anthropological data that we now possess. And even Spencer and Tylor, with all the data at their command and with all their ability to analyze and organize their essential elements, made the same mistake as Hobbes. For in the first place they made belief about the dead a result of secondary sensuous experience, instead of primary; and secondly, they made it to depend upon visual instead of olfactory experience. The sense primarily concerned in the evolution of religious aversions associated with ideas of the dead is undoubtedly that of smell. This primary aversion, by a traditional transfiguration, becomes a dread or fear of the dead and places of burial; and only when man requests of his most-used sense to show him the cause of the aversion does it become visualized. And then only is it that dreams, visions, apparitions, reflections and other illusory visual phenomena gain a superstitious meaning.

Thus it is only by misinterpretation of the act of avoiding or allaying the noisome odors of decomposition, when the real motive to the act has disappeared from view, that a people can ever explain its burial practice as a spiritual "riddance" or "aversion," or as a "laying of the ghost." For the anthropologist to accept this secondary aspect of the relation between belief and practice as being primary, and to proceed upon this assumption to the explanation of burial practices is to put the cart before the horse. Such reasoning is all of a piece with myth; it is reasoning in a circle, and will never get us anywhere in the realm of scientific knowledge.

For such reasons as I have given above, which I believe to be sound, I feel reasonably certain that my primary assumption of an obvious and constant relation between the fact of death and beliefs about the dead is justified;

that geographical conditions have played a hitherto unrecognized part in the development of burial practice and belief about the dead; that the sense of smell has had an unrecognized share in the development of religious notions and especially religious fears; that the Greek notion of an underworld abode of the dead grew out of the practice of inhumation, and that the notion of a heavenly abode of souls in like manner grew out of the practice of cremation. And it is by reason of the satisfactory corroboration of my reasoning with regard to inhumation and cremation that I suggest a primitive practice of canoe-burial on the west coast of Asia Minor—or possibly the Balkan peninsula—as the primary motive to the conception of a western abode of souls, whether as Islands of the Blest or as a continental realm of dark-browed Hades.

ORLAND O. NORRIS.

YPSILANTI, MICHIGAN.

BERNARD BOLZANO.*

(1781-1848.)

IN BOLZANO we find the virtues of human sympathy and insight coupled with the austerer virtues of the metaphysician and logician. He was a man of action as well as a man of ideas. He was well known for his kindly disposition and his broadmindedness. He possessed not only the sympathy with the poor necessary for a social reformer, but the ability to develop his ideas of social reconstruction on practical lines. Not only did he elaborate a theory of an ideal state, but he also introduced numerous reforms in the actual state of which he was a member. He studied theology very earnestly as a young man and later wrote a great deal on the subject. Even though his liberal views brought him into collision with those on whom his livelihood depended, yet he courageously continued his teaching and writing, always making it his aim to seek for truth. He was a metaphysician of some importance and his treatises on metaphysics are valuable, not only for the original thought which they contain, but also for his important criticisms of Kant. In esthetics his work is by no means without interest, and to the psychology and ethics of his day he made very valuable contributions. But preeminently he was a mathematician and logician. In his

* We regret that owing to limited time and the uncertainties of transatlantic mail service *The Monist* is compelled to go to press without receiving the author's *imprimatur*.

work on mathematical analysis and mathematical logic, he stood out from all the other thinkers of his day. He was a man of many ideas and his intellectual equipment made him able to indicate to his followers the most fruitful lines of inquiry. All through his life he worked for the good of mankind, helping it on in its search for truth.

Bernard Bolzano was born on October 5, 1781, at Prague.[1] He was the fourth son of Bernard Bolzano, an upright and philanthropic member of the Italian community at Prague. His mother was a very pious women. He had a large number of brothers and sisters, the majority of whom perished in childhood; he himself was a sickly child. In his early youth he was very much interested in mathematics and philosophy. His education was of the type usual at the end of the eighteenth century. He tells us that as a child he used to let passion completely overmaster him because he believed that he was raging not at people but at Evil itself. Bolzano was sent to one of the gymnasia of his native city, where he did not distinguish himself very much, and later proceeded to the university there. At the university he studied philosophy and subsequently theology. It was his father's wish that he should be a business man, and though his father finally gave way he showed his disapproval of his son's desire to continue his studies in various ways.

Bolzano had been brought up a Roman Catholic and he was much troubled with doubts as to whether he should take orders. Finally, however, he became convinced that difficult problems, such as the authenticity of the miracles, were not essential parts of the Catholic faith, and as in his opinion the office of priest offered the best opportunity of doing good, he took orders in 1805. At the same time he became doctor of philosophy at Prague University, and

[1] *Lebensbeschreibung des Dr. B. Bolzano mit einigen seiner ungedruckten Aufsätze und dem Bildnisse des Verfassers; eingeleitet und erläutert von dem Herausgeber* (J. M. Fesl), Sulzbach, 1836.

was appointed professor of the philosophic theory of religion.

As professor, Bolzano suffered many cramping indignities which surrounded all teachers in Roman Catholic countries at that time. To a man with Bolzano's sympathies, the position must have been a peculiarly trying one. He had a great love for young people[2] and mixed freely with the students. He was particularly sought after by the students because of his liberal views. His broad-minded interpretation of the dogmas of the Catholic faith, while provoking the distrust of the authorities, recommended him to the younger generation, and he wielded a great influence in their revolutionary schemes and was thought by many to have supported them with an enthusiasm unbecoming in a professor. At any rate, relations between Bolzano and the authorities grew more and more strained, and finally, as he would not recall what they were pleased to call his "heresies," he was dismissed on the grounds that he had "failed grievously in his duties as priest, as preceptor of religion and of youth, and as a good citizen."

After his dismissal from Prague, two ecclesiastical commissions were successively appointed by the Archbishop of Prague to inquire into the orthodoxy of his teaching. In the first commission, the majority declared that Bolzano's teaching was entirely Catholic, but the word "entirely" was deleted at the wish of the minority—which consisted of one person. This decision so enraged the obscurantist party that a large amount of evidence (not a small amount of which was "faked" for the purpose) was collected and put before the second commission. In 1822 Bolzano made two declarations in writing in which he stated that he held it "dangerous, even with the best intentions, for a man to seek and teach new points of view

[2] See A. Wishaupt, *Skizzen aus dem Leben Bolzanos*: *Beiträge zu seiner Biographie von dessen Arzte,* Leipsic, 1850, pp. 19ff.

as proofs of the truth and divine nature of the Christian Religion."[3] The commission then finally collapsed. Two years later Bolzano was pressed for a public recantation. The Archbishop of Prague brought illicit pressure to bear on him by pleading his affection for him and by declaring that a refusal would bring him to the grave. Bolzano, however, refused to recant publicly, but solemnly declared his orthodoxy in writing.

The main points of his teaching on religion are set out at some length in his *Lehrbuch der Religionswissenschaft.*[4] He defines religion as the aggregate of doctrines which influence man's virtue and happiness. He then proceeds to discuss what seemed to him the most perfect religion, viz., the Catholic faith. His reason for so regarding the Catholic faith is that it is, in his opinion, revealed by God. A religion is divinely revealed, according to Bolzano, if it is morally beneficial and if connected with it there are supernatural events which have no other use than that they serve to demonstrate this religion. In the first chapter the concepts of religion in general, and organized religion in particular, are discussed. In the third chapter he maintains that for a religion to be true it must be revealed, and then he proceeds to enunciate the characteristics of a revelation. In the second volume, he sets out to prove that the Catholic religion possesses the highest moral usefulness and that its origin has the attestation of supernatural occurrences. He discusses the evidence for Christ's miracles and the genuineness of the sources and points out the presence in Christianity of the external characteristic of revelation. He then passes on, in the third volume, to demonstrate in some detail the moral usefulness of the faith. After a discussion of the Catholic doctrine of the sources of knowledge he examines the various doctrines of the

[3] Published 1836 (Sulzbach) with autobiography.
[4] Sulzbach, 1839 (4 volumes).

Catholic church. It is interesting to notice that he regards the doctrine of the Trinity as entirely reasonable, and compares the Father to the All, the Son to humanity, and the Holy Ghost to the individual soul. In the last chapter of this volume Bolzano is concerned with the Catholic system of morals. In his investigation he discusses first Catholic ethics and then the various means of salvation recommended by the church. He examines each of the sacraments in turn.[5]

After his dismissal from Prague, Bolzano wrote a very great deal, but the internal censorship prohibited all publications in his name and even in some cases retained the manuscript. Bolzano once expressed the pious hope that some day he might be allowed to publish some work of a purely mathematical nature! After he left Prague he lived chiefly with friends at Techobuz. He came back, finally, to his native city in 1841 and continued his work with vigor until his death in 1848.

Though it was in mathematics that Bolzano did his most important work, yet in other subjects, notably in political science, his work is of considerable value. He had very great sympathy with the poor and was anxious to abolish class differences. He was convinced that the inadequacy of social organizations was the cause of poverty. He never wrote very much on the matter, but made it the subject of many of his professorial addresses. There is, however, one short manuscript[6] in which he sets out the main points of his political theory. Bolzano himself thought a great deal of this manuscript for he says in the introduction: "And small as is the number of these pages, yet the author thinks he may be allowed to attribute some value to them. Nay, he considers that this little book is

[5] For a complete list of his theological works see Bergmann, *Das philosophische Werk Bernard Bolzanos*, Halle, 1909, p. 214.

[6] "Vom besten Staate, MS. in the Royal Bohemian Museum. For a convenient summary of the MS. see Bergmann, *op. cit.*, pp. 130ff.

the best and most important legacy that he can bequeath to his fellow men if they are willing to accept it."

In Bolzano's ideal state, men and women alike are to have the privilege of voting, but a person is only allowed to vote on a matter of which he has some knowledge and in which he has some interest. Further, the right of voting is liable to forfeiture in the case of misconduct. Any citizen may put forward a suggestion. The suggestion is examined by six independent citizens, each one examining it privately, and it is only rejected if all six of the citizens reject it—and even then it is retained by the state for further reference. If it is not rejected, a general vote is taken, and if there is a majority in favor of it, it goes to a council[7] which is composed of men and women over sixty years of age, who are chosen by the people every three years. The council can only veto the decisions of the people if ninety percent of the council are against it. The government is the administrative body, its members are paid and elected by the people, and there is a strict limit to the length of time that they may remain in office. The government takes special care to prevent private individuals combining in their own interest. Bolzano looked upon war as a dreadful misfortune and in his Utopia war is only to be used as a defensive measure. Bolzano points out that internal revolutions are unlikely, for they arise in general from one of two causes—a bad constitution or poverty. Of these, poverty is to be non-existent and a revolution due to the first cause is improbable because it could only be brought about if the council opposed a change in the constitution which the people considered advisable. But the council in its wisdom would not taunt the people but would give reasons for its decision. It therefore seems unlikely that the people would rise in revolt, all the more because it is early impressed upon the young that a good

[7] The council is called the *"Rat der Geprüften."*

citizen does not work against the government, for the government's object is to work for the good of the whole state.

One of the most interesting parts of the manuscript deals with the idea of property. In the ideal state property is only desired in so far as the possession of it contributes to the common good. The only valid claim of a man to property is, therefore, that he can make it more useful to the state than any one else could. The fact that a man may possess a certain thing at a certain time is not a necessary or sufficient reason that he shall possess it altogether. The right of inheritance is not recognized. Things such as books, paintings, furniture or jewels, are given to a citizen to use but not to possess. Further, even though he may have established his claim to a certain object, yet, if at any subsequent time another citizen can make more use of it, the title of the first citizen to it is gone. Moreover, the state does not offer any compensation to a man for depriving him of anything. Thus a man whose eyesight has been cured has his glasses taken away and no compensation is made. In all the distribution of goods the government is guided entirely by the principle that the use of a certain thing should be granted to the citizen who can render it most useful to the state as a whole.

The ideals of the state are freedom and equality. There is no unequal distribution of wealth. However there is not an absolute equality of owners, for, as Bolzano points out, the possibility of increasing one's property is a powerful incitement to work. But there are limits beyond which a man cannot increase the extent of his property, and these limits are determined by the consideration of the good of the state as a while. There are "equal" right for all citizens, but the word "equal" is not to be interpreted in any narrow sense. Rather there is an adjustment between the rights of a citizen and his obligations, between his strength

and his need. The government aims at promoting religious freedom. No religion is given preferential treatment by the state. People choose their own ministers of religion and support them. But a new religion may not be preached without permission, for some might not be able to grasp all the consequences of accepting certain doctrines and beliefs. Further, a citizen may change his religion, but he must first bring proof that he has studied with earnestness the principles of the religion he is about to leave, as well as of the one which he desires to embrace.

In the education of children the special aim is the development of the mind. The teachers do not have complete freedom in the choice of what the children are taught. The Council, if it is unanimous, has the power to prevent the teaching of any particular doctrine. The children's books are censored. The censor is responsible directly to the government. And not only the children's books, but all the books in the state are censored strictly.

The question of rewards and punishments in the state is treated in a practical way. Rewards are to consist in public recognition of merit, and punishments are not arranged on a definite plan but are modified so as to suit individual cases. There is however a special proviso that no citizen is under any circumstances to be imprisoned for life.

Bolzano has some very interesting ideas on the occupations of the people in his Utopia. To begin with, the state is to support those who are not fit to work. From those who are fit, the state demands a certain fixed amount of work—the fixed amount, of course, varying from one individual to another. In return for the work the state distributes goods. Citizens are not allowed to waste their time in useless or pernicious occupations—Bolzano considered newspapers pernicious. Neither are they allowed to do things in any but the quickest and most satisfac-

tory way. Thus they are not allowed to thresh with a flail when a threshing machine has been invented, nor, presumably, to walk when there is a tram. One interesting point is that the state is to pay compensation for damage done by nature. Bad weather would quickly lose its terror for farmers in Bolzano's ideal state. Finally, those who wish to devote their lives to art or some branch of learning are supported by the state if they can produce evidence to show that it will be in the state's interest that they shall be employed in this way. The whole theory of the state is peculiarly fresh and in many respects suggestive.

But Bolzano's Utopia is only a practical illustration of his general ethical principles. The guiding principle of his inquiry may be enunciated as follows: Of all possible actions, one should always choose that one which, when all consequences have been considered, produces the greatest amount of good or the least amount of evil, for the human race as a whole, and in this estimate the good of individuals, as such, is to be left out of consideration. But Bolzano points out that if this principle is to be the highest moral law, it would be necessary to frame a definition of *good* and *bad* before any practical applications could be made. Further since he holds that an action is good if it is an action which we ought to perform, he gets back immediately to the question: What ought I to do?[8]

There then remains only the effects of action on the faculty of sensation. Bolzano argues that, since one can excite only either pleasant or unpleasant sensations and since no one would hold that it is one's duty to excite unpleasant sensations, it is obviously one's duty to excite pleasant sensations. By this process of eliminating everything except the faculty of sensation, Bolzano comes to the conclusion that the highest moral duty is the excitement

[8] For an interesting and valuable criticism of Bolzano's assertions and deductions mentioned here, see Bergmann, *op. cit.*, Part V, § 958.

of pleasant sensations. Not the least interesting part of his work in ethics is his criticism of Kant's categorical imperative. He urges the necessity for a modification in Kant's principle and points out the invalidity of Kant's theory that the opposite of a duty involves a contradiction.

Bolzano's work in esthetics is not without interest.[9] His theory of esthetics is the result, not of his own esthetic sensations, but of a painstaking analysis of the abstract idea. His definition of the scope of the subject does not make it coincide with the theory of beauty unless we include in that theory not only the sum total of truths directly concerned with beauty but also all those which stand in such a relation to them that either the former cannot be thoroughly understood without the latter or the latter without the former. To get at his concept of beauty, he eliminates goodness and attractiveness, and by this process obtains a first criterion of beauty, viz., all beauty is pleasant, i. e., it produces pleasure and this pleasure arises solely from the contemplation of the object. Further, since animals are to be excluded from esthetic enjoyment, qualities must be introduced which they do not possess, e. g., intelligence, judgment and reason. Bolzano then comes to the conclusion that it is the growth of these qualities in us that is responsible for the pleasure we find in beauty. Together with the "Ueber den Begriff des Schönen" in the Royal Bohemian Museum, there is another short treatise of Bolzano's in which a theory of laughter is elaborated.[10] Bolzano thought that laughter was caused by the rapid alternation of pleasant and unpleasant sensations and from the fact that animals and infants do not laugh he deduces that laughter is not entirely physical.[11]

In his metaphysics, Bolzano reveals himself as "one of

[9] See *Ueber den Begriff des Schönen*, Prague, 1843.
[10] *Ueber den Begriff des Lächerlichen*, 1818.
[11] See Bergmann, *op. cit.*, Part IV, § 56.

the acutest critics of the Kantian philosophy and the 'idealist' development from Fichte to Hegel."[12] He also did some important original work. His chief book on the subject,[13] entitled *Wissenschaftslehre: Versuch einer ausführlichen und grösstenteils neuen Darstellung der Logik,*[14] is divided into five sections. In the first of these he sets out to prove that objective truth exists and that it is possible for us to have knowledge of it; but he allows that in the development of the science of knowledge, which is the most fundamental of the sciences, it is necessary to use some psychological methods of treatment. In the second part, the "Theory of Elements," he treats successively ideas-in-themselves, their combination into propositions-in-themselves, the theory of true propositions-in-themselves, and finally their combination into syllogisms. He is extremely careful to distinguish between the idea-in-itself and the conceived idea. The concept of a proposition-in-itself is produced by a double abstraction. First the meaning of the proposition and the words conveying the meaning have to be separated from each other, and then one has to forget that the proposition has ever been in anybody's mind. By this means we get to the concept of a proposition-in-itself.

In the distinction that he draws between perception and conception, Bolzano himself says that he owes very much to Kant, but Bolzano disagrees with him in the use he makes of this distinction in his theory of time and space. Bolzano examines in some detail Kant's theory of time and space and his theory of the categories, making some very acute criticisms. After an investigation into the theory of the syllogism and a discussion of the function

[12] A. E. Taylor, *Mind,* October, 1915.

[13] For a criticism of Bolzano's theories see M. Palagyi, *Kant und Bolzano,* Halle, 1905.

[14] Sulzbach, 1837.

of the linguistic expression of a proposition, the "Theory of Elements" closes with a criticism of previous works on the subject. Next Bolzano considers the appearance in the mind of propositions-in-themselves. And it is in this part of his work in particular that we see the extent and depth of his learning. He treats first our subjective ideas, then our judgments, then the relation of our judgments to truth, and finally their certainty and probability. In this investigation Bolzano uses psychological methods to some extent. Then after the fourth part, the "Art of Inventing," he comes at last in the fifth part to the "Science of Knowledge Proper." The book is remarkable as much for its wealth of original thought and the clearness of expression as for the important criticisms of earlier works on the subject.

But important as is Bolzano's work in metaphysics, ethics, esthetics, and theology, it is preeminently as a mathematician that he should be remembered. Now there are two ways of looking at mathematics. One can look upon it as Huxley did: "Mathematics may be compared to a mill of exquisite workmanship, which grinds you stuff to any degree of fineness." On the other hand, one can look upon mathematics as a real and genuine science and then the applications are only interesting in so far as they contain and suggest problems in pure mathematics. From the second point of view the most important business of the mathematician is to examine and strengthen the foundations of mathematics and to purify its methods. In addition to these points of view which may be called the practical and the philosophical, a third point of view has sprung up in the last century which may be called the purely logical point of view. Whitehead describes this new point of view in the words, "Mathematics in its widest significance is the development of all types of formal, necessary, deductive,

reasoning."[15] In this purely logical system, it is proposed to treat any special development of mathematics with the help of a definite, logically connected complex of ideas, and the mathematician is not to be satisfied to solve particular problems with the help of any methods which may casually present themselves, however ingenious these methods may be. Clear definitions and unambiguous axioms must be framed and then by rigorous reasoning the theorems of the subject are to be deduced.

We find examples of the first and second points of view among the Greeks. It is said of Pythagoras that "he changed the occupation with this branch of knowledge into a real science, inasmuch as he contemplated its foundation from a higher point of view and investigated the theorems less materially and more intellectually,"[16] and of Plato that "he filled his writings with mathematical discussions, showing everywhere how much geometry there is in philosophy." Just as mathematics among the Greeks had its origin in the geometry invented by the Egyptians for practical surveying purposes, so the mathematics of the seventeenth and eighteenth century received its stimulus from the practical researches of Kepler, Newton and Laplace. But in this same fragment of Eudemus we find it recorded that Euclid tried to revise the methods used and "put together the elements, arranging much for Eudemus, finishing much for Thaetetus; he moreover subjected to rigorous proofs what had been negligently demonstrated by his predecessors."

This same work that Euclid did for Greek mathematics three hundred years B. C., the new school of nineteenth century mathematicians performed for European mathe-

[15] A. N. Whitehead, *A Treatise on Universal Algebra*, Cambridge, 1898, preface, p. vi.

[16] Extract from a fragment preserved by Proclus; generally attributed to Eudemus of Rhodes who belongs to the peripatetic school and wrote treatises on geometry and astronomy. See extracts in J. T. Merz, *History of European Thought in the Nineteenth Century*, Vol. II, p. 634.

matics. The researches of Newton had suggested a wealth of material for mathematical treatment. Newton and Leibniz had stumbled across the powerful methods of the calculus, which were of tremendous practical importance; but as Klein says, "the naive intuition was especially active during the period of the genesis of the calculus,"[17] and in the great call for powerful methods the theoretical side was almost entirely overshadowed. For example Newton assumed the existence of the velocity of a moving point at every point of its path, not troubling whether, as subsequent investigation has shown to be the case, there might not be continuous functions having no derivative. The great work then of this new school was to investigate the validity of the methods used in the two previous centuries. This was no easy task, and it is only now after one hundred years that the theory of the subject is being put on a logically satisfactory basis. The most important ideas round which the greater part of the work in mathematics centered, are those of continuity and infinity. The importance of these concepts became apparent from the work done on infinite series. A particularly simple example of series, viz., decimal fractions, was in use as early as the sixteenth century, but Leibniz was the first mathematician to have any idea of the importance of series in mathematics. Before his time it had not been realized that an infinite series can only have a meaning under certain circumstances. Unfortunately Leibniz came to the conclusion that the sum of the series

$$1 - 1 + 1 - 1 \ldots ad\ inf.$$

is $\frac{1}{2}$,[18] and so exercised a somewhat baneful influence on

[17] *Evanston Colloquium; Lectures on Mathematics delivered September, 1893*, Lecture VI.

[18] Euler in 1755 (*Instit. Calc. Diff.*) defined the sum of this series to be $\frac{1}{2}$. In the recent theory of divergent series (due in great measure to E. Borel see his *Leçons sur les séries divergentes*, Paris, 1901) one way of defining the formal sum of a divergent series Σa_n is as the limit, when it exists, of $\Sigma a_n x^n$ as x tends to unity through values less than unity. This definition has the

subsequent mathematical developments of the theory of infinite series. However it was left to the genius of Bolzano[19] to enunciate for the first time the necessary and sufficient conditions for the convergence of an infinite series. In 1804 Bolzano published his *Betrachtungen über einige Gegenstände der Elementargeometrie* (Prague), and in 1810 his *Beyträge zu einer begründeteren Darstellung der Mathematik* (Prague). In 1816 he published an important tract on the binomial theorem. In this tract his work on convergency is of great value and his investigation for a real argument (which he everywhere presupposes) is very satisfactory. Bolzano comments on the unrestricted use of infinite series which was common at the time. In 1812 Gauss had published an investigation into the circumstances under which the hypergeometric series converges, and in 1820 Cauchy delivered some extremely important lectures on analysis at the Collège de France, where he was the leader of a group of young mathematicians. Thus Bolzano, Gauss and Cauchy were the pioneers. In his book, *Der binomische Lehrsatz und als Folgerung aus ihm der polynomische und die Reihen, die zur Berechnung der Logarithmen und Exponentialgrössen dienen, genauer als bisher erweisen* (Prague), Bolzano has made a valuable criticism of earlier investigations. It is remarkable that his writings, though of great importance, received comparatively little attention at the time. According to Merz, he had not, like Cauchy, "the art peculiar to the French of refining their ideas and communicating them in

merit of simplicity and also of "consistency," i. e., When the series Σa_n converges, its sum is still the limit as x tends to unity through smaller values, of $\Sigma a_n x^n$ if this limit exists.

Defining the formal sum in this way the sum of the series $1 - 1 + 1 \ldots$ *ad* inf. is ½.

[19] Accounts of Bolzano's mathematical work were given by Otto Stolz (*Math. Ann.*, Vol. XVIII, 1881, pp. 255-279; Vol. XXII, 1883, pp. 518-519) and on pp. 37-39 of the notes at the end of the reprint of Bolzano's "Rein analytischer Beweis" of 1817 in No. 153 of *Ostwald's Klassiker.*

the most appropriate and taking manner."[20] In his *Rein analytischer Beweis* (1817) Bolzano tells us that it is very much better to publish one's mathematical work in separate treatises; in this way there is more chance of getting acute criticism. Consequently we find his mathematical work scattered about in various small treatises.[21] Also he tells us that one of his treatises had the misfortune not to be noticed by some of the learned periodicals and in others to be criticized only superficially.

In 1842, in the course of some work on the undulatory theory of light, he made a prophecy which is extremely interesting in the light of the invention of spectrum analysis and the researches of Sir W. Huggins, Kirchhoff, and others. He said: "I foresee with confidence that use will hereafter be made of it in order to solve, by observing the changes which the color of stars undergoes in time, the questions as to whether they move, with what velocity they move, how distant they are from us and much else besides." But let us return to the most important part of Bolzano's mathematical investigations.

In 1817 Bolzano published a paper we have already mentioned entitled "Rein analytischer Beweis des Lehrsatzes: dass zwischen je zwei Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reele Wurzel der Gleichung liege." This paper is, in a way, his most important work and is a triumph of careful and subtle mathematical analysis. His central theorem, as the title indicates, is as follows: If in an equation $f(x)=0$, $x=\alpha$ makes $f(x)$ positive and $x=\beta$ makes $f(x)$ negative, then there is at least one real root of the equation $f(x)=0$ between α and β. Before he begins his constructive work he criticizes very acutely the previous attempts of Lagrange and others. He points out the errors that had

[20] *Op. cit.*, Vol. II, p. 709.
[21] For complete list see Bergmann, *op. cit.*, pp. 213-214.

been made by previous investigators and he emphasizes once more the great importance of freeing mathematical analysis from the intuitional treatment to which it had formerly been subjected. In order to prove his main theorem, Bolzano found it necessary to introduce the concept of the continuity of a function, the notion of the upper limit of a variate and some important work on infinite series. His method is briefly as follows:

1. He introduces the concept of "continuity." A function is said to be "continuous" for the value x if the difference between $f(x+\omega)$ and $f(x)$ can be made less than any assigned number, however small, if only ω is taken sufficiently small.

2. He discusses the convergence of infinite series and makes the following important statement. "If the difference between the value of the sum of the first n terms and the first $n+r$ terms of a series can be made as small as we please, for all values of r, if only we take n large enough, then there is one number X and only one such that the sum of the first p terms approaches ever more and more nearly to X as p increases." Unfortunately his proof of this theorem is not rigorous and his discussion only renders the existence of X probable.

3. From his work on infinite series Bolzano passes on to an extremely important theorem in which he introduces the new idea of an upper limit. And the theorem, as it occurs in this paper, gains in importance from the fact that the method used is one of fundamental importance in analysis. The theorem runs as follows: "If u_n be such a number that the property M holds for all values of x which are less than u_n, and if the property does not hold for all values of x without exception, then of all the numbers u_n satisfying this condition there is one (say U) which is greater than all the others." This theorem, which might appear obvious to those who allow their geometrical in-

tuitions to cloud their mathematical ideas, is proved by Bolzano with great care and completeness. The method used in the proof was used a great deal by Weierstrass and is now known as the "Bolzano-Weierstrass" process. As the method is of such great importance, we will indicate roughly the way it is used in the proof of this theorem. It will be convenient to call x's which have the property M "suitable" x's and x's which do not have the property M "unsuitable" x's; and further to call a number N a "suitable" number if all x's which are less than N have the property M, and to call a number N an "unsuitable" number if there are some values of x, less than N, which do not have the property M. Now it is obvious that there is a positive number D, such that $u_n + D$ is an unsuitable number. Then, bisecting the interval between u_n and $u_n + D$, we get the number $u_n + D/2$; bisecting the interval between u_n and $u_n + D/2$ the number $u_n + D/2^2$; and so on. When either all the numbers $u_n + D/2^r$ for $r =$ 1, 2, 3... are unsuitable or there is a number R such that $u_n + D/2^R$ is an unsuitable and $u_n + D/2^{R-1}$ a suitable number. In the first case the existence of U is established, U being equal to u_n. In the second case we repeat the process, dividing the interval between $u_n + D/2^{R-1}$ and $u_n + D/2^R$. Again, either all the numbers $u_n + D/2^R + D/2^{R+s}$, $s = 1, 2....$ are unsuitable or there is a number S such that $u_n + D/2^R + D/2^{R-S}$ is an unsuitable and $u_n + D/2^R + D/2^{R+S} - 1$ a suitable number. We continue the same process: if it does not terminate we get finally to an infinite series

$$u_n + D/2^R + D/2^S + D/2^T + \ldots$$

and since R, S, T... are positive integers the series obviously satisfies the conditions of the theorem in paragraph (2) above, and so there is a definite limit to which it tends.

this limit being the "upper limit" U in question. The existence-theorem for an upper limit is thus established.

4. Bolzano next attacks the following theorem: "$f(x)$ and $\varphi(x)$ are continuous functions of x and for $x = \alpha$, $f(x) < \varphi(x)$ and for $x = \beta$, $f(x) > \varphi(x)$: then there is a value of x between α and β for which $f(x) = \varphi(x)$." We will indicate the method Bolzano uses to prove it and we shall see exactly why he found it necessary to establish the existence of an "upper limit." Bolzano shows that, since $f(x)$ and $\varphi(x)$ are continuous, there is a number ω such that all numbers less than it satisfy the relation $\varphi(\alpha + \omega) > f(\alpha + \omega)$. Such a number we may call as in paragraph (3) a "suitable" number. Then from a direct application of the theorem about an upper limit he establishes the existence of an upper limit, say U, for all suitable numbers. It is then easy to show that $f(\alpha + U)$ cannot be less than $\varphi(\alpha + U)$ and cannot be greater than $\varphi(\alpha + U)$ and is therefore equal to $\varphi(\alpha + U)$. In this kind of way Bolzano proves the existence of the value of x between α and β giving $f(x) = \varphi(x)$.

5. Finally Bolzano proves that an expression of the form

$$a + bx^m + cx^n + \ldots + px^r,$$

in which $m, n, \ldots r$ are positive integers, is continuous. Then by means of an easy application of a slightly modified form of the theorem in (4) he proves that there is at least one real root between α and β. The whole paper is extremely valuable and it is interesting to see how Bolzano was led from his central theorem to the theorem in (4), to the concept of "continuity" and the idea of an "upper limit," and in the existence-theorem for the upper limit to the question of the convergence of series.

In mathematical logic and in the theory of infinite numbers, Bolzano's work was also of great importance. Bol-

zano's definition of the continuum is of some interest in itself. He defines a continuum as a set of points such that every point has another point also belonging to the set as near to it as we please.[22] This is expressed in modern phraseology by saying that the continuum is a set of points which is "everywhere dense." The name continuum is now used (after Cantor) only for a set of points which is not only "everywhere dense" but also "perfect." A set of points is "perfect" when every convergent sequence has a limit which is itself a number belonging to the set, and conversely when every number is the limit of properly chosen convergent sequences of numbers themselves belonging to the set.[23] Thus Bolzano would call the set of rational numbers a "continuum," but this set is not perfect and is therefore not a "continuum" in the modern sense of the word. In his work on infinite numbers Bolzano anticipated to some extent the work of Georg Cantor. An "infinite" collection is defined to be a collection which has no last term.[24] He proves that the number of natural numbers and the number of real numbers is infinite, and he sees (§ 49) that the number of these two collections is different. Bolzano also recognizes the fact that it is possible to arrange the points in two lines of different lengths so that each point of one collection corresponds to one single point of the other collection and *vice versa,* no point being left without a corresponding point. This brilliant idea of a one-one correspondence went a long way toward dispersing the cloud of mystery which hung over the contemporary infinite number. Leibniz had stated the difficulty quite plainly. Every number can be doubled, he said, therefore the number of natural numbers and the number of even natural numbers is the same. Therefore the whole

[22] *Paradoxien des Unendlichen,* Leipsic, 1851, 2d ed., Berlin, 1889, § 38.

[23] See E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series,* Cambridge, 1907, p. 49.

[24] *Paradoxien des Unendlichen,* § 9.

is equal to the part—which is absurd. Bolzano realized that there is no real contradiction in this. This same idea of the one-one correspondence between points belonging to certain sets of points has led to the modern idea of "reflexiveness" of infinite numbers. The property of "reflexiveness"[25] together with that of "non-inductiveness,"[26] which disposes of all attempts to count up infinite collections or identify the number of terms in an infinite collection with the ordinal number of the last, has removed all serious difficulties and has helped to make it possible to put the concept of an infinite number on a logical foundation.[27] Defining "similar" classes as classes whose terms have a one-one relation to each other and the "cardinal number" or "power" of a class as the class of all similar classes, we see immediately that the class of natural numbers and the class of even natural numbers have the same cardinal numbers. Thus Bolzano was quite right in seeing no contradiction in Leibniz's statements.

From these few references to isolated theorems and statements in Bolzano's work, it is seen that he had most of the ideas essential in the modern view of mathematics, and that in mathematics at least Bolzano's work has been a source of inspiration to those who came after him. Whether in his theology, his ethics, his political science, his metaphysics, or his mathematics, the desire for clearness of concepts was always his aim. Even the parts of his work which are no longer of intrinsic interest, e. g., his esthetics or his theory of laughter, have an interest for us in that they show us the methods he used in seeking

[25] A number is said to be "reflexive" if it is not increased by adding one to it. See B. Russell, *Our Knowledge of the External World as a Field for Scientific Method in Philosophy,* Chicago and London, 1914, p. 190.

[26] A number is said to be "non-inductive" if it does not possess deductive properties. See B. Russell, *op. cit.,* p. 195.

[27] Cf. the definitions "that which cannot be reached by mathematical induction starting from 1" and "that which has parts which have the same number of terms as itself," B. Russell, *The Principles of Mathematics,* Cambridge, 1903, Vol. I, p. 368.

for truth. That there is objective truth and that we can have knowledge of it—this was the thesis which he set before him in his work. In mathematics especially his work was needed, for whereas idealists maintained that mathematics deals only with appearances, empiricists insisted that mathematics could only approximate to the truth. Bolzano's life work was to start mathematicians on the right way to refute both the idealists and the empiricists. His method of strictly logical analysis of the ideas of continuity and the infinite was the clue which was followed up by all the great mathematical logicians and mathematical analysts of the nineteenth century, until finally the fundamental thesis has been proved that all concepts of pure mathematics are wholly logical. Thus Bolzano was one of the first to suspect and in this he was a worthy successor of the great Leibniz. Unlike most mathematicians of his day, Bolzano did not in his thirst for results succumb to D'Alembert's maxim, *Allez en avant, la foi vous viendra.*

We live in days when some of the contradictions and paradoxes which have perplexed the human race since the days of Zeno are being finally cleared up. Do not let us forget the work of Bolzano who with painstaking endeavor sowed the seeds of this great revolution in mathematical ideas.

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A MEDIEVAL INTERNATIONALIST.

ARBITRATION, a league of peace and a council of conciliation seem to be very modern suggestions as methods of avoiding war between civilized nations. Some hints of these, however, can be found in Kant's *Perpetual Peace* and in the *grand dessein* as expounded by the Abbé de S. Pierre. These schemes belong to the Revolutionary and Renaissance periods. But even before, in the Middle Ages, similar schemes are to be found in the work of Petrus de Bosco (Pierre Dubois).

The political acuteness of this brilliant thinker can only be understood by allowing for the fact that he had listened at Paris to "that most prudent friar Thomas Aquinas"[1] and by remembering that he wrote while the official politicians were engineering war after war for no purpose. His work on international politics is contained in the unprinted *Summaria brevis.....abbreviationis guerrarum* and in the *"De recuperatione Terre Sancte,"* published (1891) in the *Collection des Textes*. I propose to summarize and comment upon the latter, not as of merely archeological interest, but as an early attempt to grapple with the same political problem which we now face.

The treatise is supposed to deal with a plan for recovering the Holy Land and is addressed in 1306 to Edward I, "King of England and Scotland, Lord of Ireland and Duke of Aquitaine," as a great legislator and one who was

[1] Par. 63, *De recup. Terre Sancte.* (In medieval Latin final *æ* becomes *e*.)

specially interested in a new crusade. But this initial purpose of the treatise, even if it was intended by the author as more than a mere *captatio benevolentiae,* is certainly subordinated to the general problem of international policy among the European states.[2] The order of the argument is confused, the author continually going back to a subject after he has left it for some other. He writes well, but too eagerly to be as exact as the philosophers of his day. He is genuinely excited by the pressing importance of establishing peace. I shall, therefore, not follow the order of the treatise, but state first the nature of the problem as it appears to Dubois and then his suggestions for solution.

War between European countries and kings, says Dubois, is the chief hindrance to "having time for progress in morality and knowledge." War breeds war until war becomes a habit.[3] The deaths of one war cause speedy preparations for revenge.[4] "We should seek a general peace and pray God for it, that by peace and in time of peace we may progress in morality and the sciences, since we cannot otherwise; as the Apostle feels when he says: 'The peace of God which passeth all understanding keep your hearts and your minds:' your minds, which are reasonable souls, are not kept but are often destroyed by wars, discords and civil brawls which are like wars, and by the continuance of all such. Therefore, as far as he can, every good man should avoid and flee them; and when he takes to war, being unable otherwise to obtain his rights, he ought as much as possible to shorten it....Thus universal peace is the end we seek."[5]

[2] Guillaume de Nogaret uses the same pious cover for his scheme of social reform. One had to bow, so to speak, to the crusading ideal and then one was free to suggest anything!

[3] Quanto frequentius bella committunt, tanto magis appetunt committere, hoc consuetudine magis quam emendatione deputantes." Par. 2.

[4] "Ad bellum et vindictam voluntariam se preparant."

[5] Par. 27, *in fine.*

It is agreed that peace is desirable; but, says Dubois, "since it is proved that neither the Scriptures, nor sermons drawn from the Scriptures, nor the elegant lamentations and exhortations of preachers have been sufficient to stop frequent wars and the temporal and eternal death of so many human beings which have resulted, why should there not be found at last a new remedy for militarism (*remedium manus militaris*), as for example a judiciary backed by force (*justicia necessario compulsiva*)?" (par. 109). "This is an argument," he declares, "to which a reply is impossible morally and politically speaking." Peace has come within states by *vis coactiva*: so also it will come between states. One could not have a clearer statement of political judgment upon the evidence. The author himself says that he depends upon experience for his opinions: and he declares that exhortations to peace and praise of its excellencies and even rhetorical attacks on war are politically valueless. They have been tried and they have failed.

Before speaking, however, of the means by which peace is to be established between states, we must notice the plan which is *not* suggested by Pierre Dubois. The governing ideal of medieval politics, unity, led many to look for peace through subordination to one overlord. "Now there is no sane man, I think," Dubois writes, (par. 63), "who could think it likely that in this latest age (*in hoc fine saeculorum*) there could be one monarch of the whole world in temporal affairs who would rule all and whom as superior all would obey. For, if there were any attempt at this there would be wars, seditions and discords without end; nor would there be any one who could allay them by reason of the number of different nations, the distance and distinction between countries and the natural inclination of men to diverge. Although some have been popularly called "lords of the world" nevertheless I think that since the countries were settled there never has been any one

whom all obeyed." That passage, if it seems to condemn Dante as a *homo non sane mentis,* certainly shows an historical acumen and a political judgment far superior to the opinions of the *De Monarchia.* Dubois recognizes the impossibility of arriving at peace by means of the conquest by one state of all other states. He sees that world-power is nonsense.

It must be admitted, however, that from the passages of the *Summaria brevis* which have been commented upon by M. de Wailly and Ernest Renan, one might judge that Dubois hoped for a domination in Europe of the French king. He held, indeed, that it should be arrived at by diplomacy and not by war, but in the above passage of the *De recuperatione* he seems to condemn not merely any special means, but dreams of domination by a single lord.

Inconsistency may be urged against him, and yet it must be remembered that here he is writing to the English king and also that he may very well have felt uncertain as to how the *vis coactiva* above the warring states might be established, even if he held quite clearly to the notion that the ultimate supremacy of one monarch was impossible. But let us turn to the definite political means he suggests for establishing peace between European states.

The means by which such peace is to be arrived at are: *First*: International arbitration and the establishment of an international judiciary. This is to begin by a general council (par. 3), a preliminary to all medieval and early Renaissance plans for reform. But what is unusual in Pierre Dubois is the statement that the difficulty of arranging matters is due to the fact that the cities of Italy, for example, and the various princes acknowledge no superior. "Before whom then," he asks, "can they bring their disputes? It can be answered that the council should establish elected arbiters (*arbitros*)[*] religious or others, prudent, experienced and trustworthy men." These are to select

three prelates and three others for either party to the dispute. They are to be well paid and such as are not likely to be corrupted by affection, hate, fear, greed or otherwise. They are to meet at a suitable place, to have presented to them in a summary and clear form, without minor and unimportant details, the pleas of either side. They are to take evidence from witnesses and documents, each witness being examined by at least two trustworthy and careful members of the "jury." The depositions are to be written and preserved. "For the decision, if it is expedient, they are to have assessors (*assessores*) who are thought by them most trustworthy and best trained in the divine, the canon and the civil law."[6]

Secondly, these decisions must be made effective. The Holy See is recognized as an influnce, but excommunication had better not be used. "Temporal punishment, although incomparably less than eternal, will be more feared."[7] The suggestions in detail of Pierre Dubois are perhaps a little comic, but we must allow for the situation. In the first place any group making war shall, after the war is over, be removed bodily and sent to colonize the Holy Land! If they do not oppose the movement, they may take some of their property with them. The author feels that it may be difficult. He then goes on as to other measures. Suppose, he says, that the Duke of Burgundy declares war against the King of France,—the king should then institute an economic boycott[8] and by a general council the same boycott should be declared by all Europe. Active military measures should be taken to devastate the country so that the whole people should feel it: Dubois, it seems, would adopt extreme measures to prevent war spreading, his main

[6] Par. 12, *De recup. Terre Sancte.*

[7] Excommunication is to be used (§ 101) but not depended upon by itself. Any one refusing to enter the league of peace (*pacis universalis federa*) is to *be immediately attacked.*

[8] Prohibebit quod nullus ad terras eorum deferat victualia, arma, merces et alia quaecumque bona, etiam quacumque causa sibi debita," (par. 5).

point being that in whatever corner it broke out the whole of Europe should act together and at once to stop it.

The reader may feel that this is hopelessly unpractical, since we could not act thus against any great country or against any combination of countries. But we must remember (1) that Dubois supposes Europe to be one political system (*respublica Christicolarum*) able to act in concert at least in some issues, and (2) that every war begins, according to him, in some comparatively small group. Thus practically, if Europe had adopted strong economic, even without military, action during the Balkan wars of 1912 and 1913, the war of 1914 might never have occurred. And surely it is not unpractical to suggest that all civilized countries should act together in the case of any conflict breaking out such as that of 1912. Deal effectively with the small conflicts and the first difficulty is met with regard to the larger. But one can imagine the horror of medieval diplomatists if all the states were asked to prevent any small wars by direct intervention of enforcing arbitration. Even to-day all the schemes for rearranging international politics start from the present almost universal war. I cannot help feeling, however, that Dubois was right. Our schemes for doing without war must inculcate combined action in *small* wars. Deal effectively with them and we may never have to deal at all with war between great states. It is the spark, not the conflagration, that we must consider first: and perhaps European diplomacy was more futile in 1912 than in July 1914, although the results of inaction did not show themselves till August, 1914. But let us return to the general thesis and omit further applications of it.

After details as to raising funds for a common force and plans for a common advance on the Holy Land, Dubois recalls himself to his main interest, "a general peace." In the *third* place therefore, he says that no external measures will be effective until the religious attitude is changed.

This opens an elaborate project for the reform of the Roman church. Dubois says (par. 29) if the pope really wants to stop war "he must begin with his brothers the cardinals and his fellow bishops." They are always going to war (*ipsi guerras movent*). Their attitude is quarrelsome even in England and France where they do not actually fight. The monks are as bad. But the whole attack is common to many writers of the date of Pierre Dubois. His remedies are extreme. First he suggests that if the pope had no "temporal power," no one need to go to war for him and that would be a beginning; and next, he actually proposes the confiscation of ecclesiastical property by states and the use of the wealth for common European civilization![9] But how?

The *fourth* suggestion of Pierre Dubois is that the money should be spent in education.[10] The purpose of the education, according to the general thesis as to the taking of the Holy Land, is directed by the general need of non-military contact with the East. It is urged that you can only hold the East effectively by intellectual superiority to it.

Then begins a long and elaborate scheme of education, primary and secondary. University education is implied but not dealt with in detail. All this is to occur in the Holy Land. It is a well-known medieval trick for writing a Utopia. In 1223 "The Complaint of Jerusalem" gave a plan for reconstructing European society under the guise of a scheme for an Eastern kingdom. So here Dubois, appearing to speak of what ought to be done when the Holy Land is established as a state, is really speaking of the remedies which ought to be applied in Europe. In the matter of education he is as original as in politics, but what is most interesting to us now are the hints for bringing

[9] Par. 57. "Que tendit ad reformationem et unitatem veram totius reipublice catholicorum."

[10] Par. 60. "Studentes et eorum doctores vivent de bonis dictorum prioratum, etc."

the European nations together. Colleges for boys and for girls are to be established where "modern languages" are to be taught—"the literary idioms, especially of Europe, that by these scholars trained to speak and write the languages of all, the Roman church and the princes of Europe should be able to communicate with all men." Some are also to be taught medicine, some surgery—the girls also (par. 61); and these girls, in the medieval fashion perhaps, are to be married to foreigners, even Orientals (*ditioribus Orientalibus in uxores dari*). I need not detail the plans for intermarriage and colonization, among which is inserted a suggestion for a married clergy (par. 102). A long section follows upon the utility of scientific knowledge "according to brother Roger Bacon" (par. 79) and upon the variety of human knowledge in general. There are interesting hints as to the transformation of convents into girls' schools, and as to military reform, for example the institution of definite uniforms (par. 16). But all these do not bear directly upon his plans for peace and we may therefore omit them here. His boldness of conception is clear.

The other element in his Utopia, which is to establish peace, is a modification of the processes of law (par. 90 f.). The processes must be shortened according to a definite plan; but the detail need not concern us here. The fact remains that he saw that social, educational and religious reform within the state are all means for the attainment of international peace.

The closing section of the work (110-142) are addressed to Philip, king of France, who is asked to send the preceding to Edward I. Dubois urges the economic gain from the abolition of wars, and in the meantime the institution of various military reforms—as for example the regular payment of troops. It is amusing to note that the author feels the danger to himself from the powers that be, if his projects are made too public. He therefore asks

both Edward I and Philip to consider his ideas more or less privately; and he hints that one who does not happen to hold popular opinions may suffer even physical assault.

So far as we know nothing evil happened to Pierre Dubois. He was a lawyer who worked first for the king of France and afterward, when he wrote the *De recuperatione,* in the service of Edward I in Guyenne. He seems to have represented the central government in either case, and to have found his chief opponents among the churchmen. He is known as the author of a popular pamphlet in French against papal claims, as the writer of a few short Latin treatises, and as the elected representative of Coutances at the *Etats Généraux* which met in Tours in 1308. After that nothing is known of him.

More than six hundred years have gone since the treatise of Pierre Dubois was forgotten: and one may well rub one's eyes in wonder at what is now occurring in Europe. Perhaps we are dreaming. The practical man will say that the old plans for political reform are by current events proved to be valueless; that the internationalists are shown by the facts to be unable to understand real politics. And yet one would have thought that any plan might have been better worth trying than one which has brought us to our present pass. However that may be we should not despair too soon. Ecclesiastical reformation was suggested for hundreds of years before Europe arrived at the comparatively tolerant situation in religion now established. Political reformation may be more difficult, but the work of its forerunners is important. *Si Lyra non lyrasset, Lutherus non saltasset*: so also in politics, the effective reformer is taught by his predecessors who found the circumstances of their time too strong for them.

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CLASS, FUNCTION, CONCEPT, RELATION.[1]

IN my *Grundlagen der Arithmetik* of 1884 I have tried to make it seem probable that arithmetic is a branch of logic and need not borrow any ground of proof whatever from experience or intuition. The actual demonstration of my thesis is carried out in my *Grundgesetze* of 1893 and 1903 by the deduction of the simplest laws of numbers by logical means alone. But to make this proof convincing, considerably higher claims must be made for deduction than is habitually done in arithmetic.[2] A set of a few methods of deduction has to be fixed beforehand, and no step may be taken which is not in accordance with them. Consequently, when passing over to a new judgment we must not be satisfied, as mathematicians seem nearly always to have been hitherto, with saying that the new judgment is evidently correct, but we must analyze each step of ours into the simple logical steps of which it is composed, —and often there are not a few of these new steps. No hypothesis can thus remain unnoticed. Every axiom which is needed must be discovered, and it is just the hypotheses which are made tacitly and without clear consciousness that hinder our insight into the epistemological nature of a law.

In order that such an undertaking be crowned with success, the concepts which we need must naturally be con-

[1] [Translated from the *Grundgesetze der Arithmetik* by Johann Stachelroth and Philip E. B. Jourdain.]

[2] *Grundlagen*, pp. 102-104.

ceived distinctly. This is true especially in what concerns the thing that mathematicians denote by the word "aggregate" (*Menge*). It seems that Dedekind, in his book *Was sind und was sollen die Zahlen?*[3] of 1888, uses the word "system" to denote the same thing. But in spite of the exposition which appeared four years earlier in my *Grundlagen,* a clear insight into the essence of the matter is not to be found in Dedekind's work, though he often gets somewhat near it. This is the case in the sentence:[4] "Such a system S is completely determined if of everything it is determined whether it is an element of S or not. Hence the system S is the same as the system T (in symbols S = T) if every element of S is also element of T and every element of T is also element of S." In other passages, on the other hand, Dedekind strays from the point. For instance:[5] "It very frequently happens that for some reason different things *a, b, c,*... can be considered from a common point of view, can be put together in the mind, and we then say that they form a *system* S." Here a presentiment of the correct idea is contained in the words "common point of view"; but the "putting together in the mind" is not an objective characteristic. In whose mind, may I ask? If they are put together in one mind and not in another do they then form a system? What is to be put together in my mind must doubtless be in my mind. Then do not things outside myself form systems? Is a system a subjective formation in each single soul? Is then the constellation Orion not a system? And what are its elements? The stars, the molecules, or the atoms? The following sentence[6] is remarkable: "For uniformity of expression it is advantageous to admit the special case that a system S is composed of a *single* (one and one only) element *a*: the thing *a* is an element of S, but every thing

[3] [English translation under the title *Essays on the Theory of Numbers,* Chicago and London, 1901. See especially p. 45.]

[4] [*Ibid.*, p. 45.] [5] [Ibid.] [6] [*Ibid.*]

different from a is not an element of S." This is afterward[7] understood in such a way that every element s of a system S can be itself regarded as a system. Since in this case element and system coincide, it is here quite clear that, according to Dedekind, the elements are the proper constituents of a system. Ernst Schröder in his lectures on the algebra of logic[8] goes a step in advance of Dedekind in drawing attention to the connection of his systems with concepts, which Dedekind seems to have overlooked. Indeed, what Dedekind really means when he calls a system a "part" of a system[9] is that a concept is subordinated to a concept or an object falls under a concept. Neither Dedekind nor Schröder distinguish between these cases because of a mistake in point of view which is common to them both. In fact, Schröder also, at bottom, considers the elements to be what really make up his *class*. An empty class should not occur with Schröder any more than an empty system with Dedekind. But the need arising from the nature of the matter makes itself felt in a different way with each writer. Dedekind says:[10] "On the other hand, we intend here for certain reasons wholly to exclude the empty system, which contains no element at all, although for other investigations it may be convenient to invent (*erdichten*) such a system." Thus such an invention is permitted; it is only desisted from for certain reasons. Schröder dares to invent an empty class. Apparently then both agree with many mathematicians in holding that we may invent anything we please that does not exist,—even what is unthinkable; for if the elements form a system, then the system is annulled at the same time as the ele-

[7] [*Ibid.*, p. 46.]

[8] *Vorlesungen über die Algebra der Logik* (*exakte Logik*), Vol. I, Leipsic, 1890, p. 253. [This reference of Frege seems wrong and it should perhaps rather be to such a page as p. 100. Cf. also Frege's later critical study: "Kritische Beleuchtung einiger Punkte in E. Schröders Vorlesungen über die Algebra der Logik," *Archiv für systematische Philosophie*, Vol. I, 1895, pp. 433-456.]

[9] [*Op. cit.*, p. 46:]

[10] [*Ibid.*, pp. 45-46.]

ments. As to where the limits of this license lie and whether indeed there are any such limits, without any doubt we will not find much clearness and agreement;—and yet the correctness of a proof may depend on such questions. I believe I have settled them in a way that is final for all intelligent persons, in my *Grundlagen*[11] and in my lecture "Ueber formale Theorien der Arithmetik."[12] Schröder invents his zero-class and thus gets into difficulties.[13] We do not find, then, a clear insight into the matter with either Schröder or Dedekind; but still the true position of affairs is seen whenever a system is to be determined. Dedekind then brings forward properties which a thing must have in order to belong to a system, i. e., he defines a concept by its characteristics.[14] If now a concept is made up of characteristics and not of the objects falling under the concept, there are no difficulties to be urged against an empty concept. Of course in this case an object (*Gegenstand*) can never also be a concept, and a concept under which only one object falls must not be confused with this object. Thus we are finally left with the result that the number datum contains an assertion about a concept.[15] I have traced back number to the relation of similarity[16] (*Gleichzahligkeit*) and similarity to univocal correspondence (*eindeutige Zuordnung*). Of "correspondence" much the same holds as of "aggregate" (*Menge*). Nowadays both words are often used in mathematics, and

[11] Pp. 104-108.

[12] *Sitzungsberichte der Jenaischen Gesellschaft für Medicin und Naturwissenschaft*, July 17, 1885.

[13] Cf. E. G. Husserl, *Göttinger gelehrte Anzeigen*, 1891, No. 7, p. 272,—where, however, the difficulties are not solved.

[14] On *concept, object, property,* and *characteristics*, cf. my *Grundlagen*, pp. 48-50, 60-61, 64-65, and my essay "Ueber Begriff und Gegenstand," *Vierteljahrsschrift für wissenschaftliche Philosophie*, Vol. XVI, 1892, pp. 192-205.

[15] See *Grundlagen*, pp. 59-60.

[16] [The same idea and word were used by Dedekind (*op. cit.*, p. 53); and the same idea but with the name "equivalence" was used by Georg Cantor (cf. *Contributions to the Founding of the Theory of Transfinite Numbers*, Chicago and London, 1915, pp. 40, 86).]

very often there is lacking an insight into what is intended to be denoted by them. If my opinion is correct that arithmetic is a branch of pure logic, then a purely logical expression has to be chosen for "correspondence." I choose the word "relation." *Concept* and *relation* are the foundation stones upon which I erect my structure.

But even when concepts have been grasped quite precisely, it would be difficult—nearly impossible in fact—to satisfy the demands we have had to make of a process of proof without some special means of help. Now such a means is my ideography (*Begriffsschrift*), the explanation of which will be my first problem. The following remarks may be noticed before we proceed farther. It is not possible to define everything, hence it must be our endeavor to go back to the logically simple which as such cannot properly be defined. I must then be satisfied with referring by hints to what I mean. Before all I have to strive to be understood, and therefore I will try to develop the subject gradually and will not attempt at first a full generality and a final expression. The frequent use made of quotation marks may cause surprise. I use them to distinguish the cases where I speak about the sign itself from those where I speak about its denotation. Pedantic as this may appear, I think it necessary. It is remarkable how an inexact mode of speaking or writing which perhaps was originally employed only for greater convenience or brevity and with full consciousness of its inaccuracy, may, when that consciousness has disappeared, end by confusing thought. Has it not happened that number signs have been mistaken for numbers, names for the things named, the mere auxiliary means for the real end of arithmetic? Such experiences teach us how necessary it is to make the highest demands of exactitude in manner of speech and writing. And I have taken pains at least to do justice to such demands wherever it seemed to be of importance.

If[17] we are asked to give the original meaning of the word "function" as used in mathematics, we easily fall into saying that a *function* of x is an expression formed by means of the notations for sum, product, power, difference, and so on, of "x" and definite numbers. This attempt at a definition is not successful because a function is here said to be an *expression,* a combination of signs, and not what the combination stands for. Then probably another attempt would be made with "denotation (*Bedeutung*) of an expression" instead of "expression." But there appears the letter "x" which indicates a number, not as the sign "2" does, but indefinitely. For different number-signs which we put in the place of "x", we get, in general, different denotations. Suppose for example, that in the expression "$(2+3.x^2)x$", instead of "x" we put the number-signs "0", "1", "2", "3", one after the other; we then get as corresponding denotations the numbers 0, 5, 28, 87. Not one of these denotations can claim to be our function. The essence of the function is in the correspondence that it establishes between the numbers whose signs we put for "x" and the numbers which then appear as denotations of our expression,—a correspondence which is represented to intuition by the course of the curve whose equation is, in rectangular coordinates, "$y=(2+3.x^2)x$". In general, then, the essence of the *function* lies in the part of the expression which is outside the "x". The expression of a function *needs completion* (*ist ergänzungsbedürftig*) and is *not satisfied* (*ungesättigt*). The letter "x" only serves to keep places open for a numerical sign which is to complete the expression, and thus makes known the special kind of need for completion that constitutes the peculiar nature of the function indicated above. In what follows,

[17] Cf. my lecture *Funktion und Begriff,* Jena, 1891, and my essay "Ueber Begriff und Gegenstand" cited above. My *Begriffsschrift* of 1879 now no longer represents my standpoint, and thus should only be used with caution to illustrate what I said here.

the Greek letter "ξ" will be used[18] instead of the letter "x". This keeping open is to be understood in this way: All places in which "ξ" stand must always be filled by the same sign and never by different ones. I call these places *argument-places* and that whose sign or name takes these places in a given case I call *argument of the function* for this case. The function is completed by the argument; I call what it becomes on completion the *value* of the function for the argument. We thus get a name of the value of a function for an argument when we fill the argument-places in the name of the function with the name of the argument. Thus, for example, "$(2+3.1^2)1$" is a name of the number 5, composed of the function-name "$(2+3.\xi^2)\xi$" and "1". The argument is not to be reckoned in with the function, but serves to complete the function which is unsatisfied by itself. If in the following an expression like "the $\Phi(\xi)$" is used, it is always to be observed that the only service rendered by "ξ" in the notation of the function is that it makes the argument-places recognizable; it does not imply that the essence of the function becomes changed when any other sign is substituted for "ξ".

To the fundamental operations of calculation mathematicians added, as function-forming, the process of proceeding to the limit as exemplified by infinite series, differential quotients and integrals; and finally the word "function" was understood in such a general way that the connection between value of function and argument was in certain circumstances no longer expressed by signs of mathematical analysis, but could only be denoted by words. Another extension consisted in admitting complex numbers as arguments and consequently also as function-values. In both directions I have gone still farther. While, indeed, the

[18] Nothing, however, is fixed by this for our ideography. The "ξ" never appears in the developments of the ideography itself, and I only use it in my exposition of it and in illustrations.

signs of analysis were hitherto on the one hand not always sufficient, they were on the other hand not all employed in the formation of function-names. For instance, "$\xi^2=4$" and "$\xi>2$" were not allowed to count as names of functions; but I do so allow them. But that indicates at the same time that the domain of function-values cannot remain limited to numbers; for if I take as arguments of the function $\xi^2=4$ the numbers 0, 1, 2, 3, in succession, I do not get numbers. I get: "$0^2=4$", "$1^2=4$", "$2^2=4$", "$3^2=4$", which are expressions of one true and some false thoughts. I express this by saying that the value of the function $\xi^2=4$ is either the "*truth-value* (*Wahrheitswerth*) of the true or of the false."[19] From this it can be seen that I do not intend to assert anything by merely writing down an equation, but that I only designate (*bezeichne*) a truth-value, just as I do not intend to assert anything by simply writing down "2^2" but only *designate* a number. I say: "The *names* "$2^2=4$" *and* "$3>2$" *denote* the same truth-value" which I call for short *the true*. In the same manner "$3^2=4$" and "$1>2$" *denote* the same truth-value, which I call for short *the false* just as the name "2^2" *denotes* the number 4. Accordingly I say that the number 4 is the "*denotation*" of "4" and of "2^2", and that the true is the "denotation" of "$3>2$". But I distinguish the "meaning" (*Sinn*) of a name from its "denotation" (*Bedeutung*). The names "2^2" and "$2+2$" have not the same *meaning*, nor have "$2^2=4$" and "$2+2=4$". The meaning of the name of a truth-value I call a "*thought*" (*Gedanken*). I say further that a name "*expresses*" (*ausdrückt*) its meaning and "*denotes*" its denotation. I "*designate*" (*bezeichne*) by a name what it means.

The function $\xi^2=4$ can thus have only two values, the

[19] I have shown this more exhaustively in my essay "Ueber Sinn und Bedeutung" in the *Zeitschrift für Philos. und phil. Kritik*, Vol. C, 1892, pp. 25-50).

true for the arguments $+2$ and -2 and the false for all other arguments.

Also the domain of what is admitted as argument must be extended,—indeed, to objects quite generally. *Objects* (*Gegenstände*) stand opposed to functions. I therefore count as an object everything that is not a function; thus, examples of objects are numbers, truth-values, and the *ranges* (*Werthverläufe*) to be introduced further on. The names of objects—or *proper names*—are not therefore accompanied by argument-places, but are *satisfied* like the objects themselves.

I use the words, "the function $\Phi(\xi)$ has the same *range* as the function $\Psi(\xi)$", as denoting the same as the words, "the functions $\Phi(\xi)$ and $\Psi(\xi)$ have the same value for the same argument." This is the case with the functions $\xi^2=4$ and $3.\xi^2=12$, at least if numbers are taken as arguments. But we can also imagine the signs of evolution and multiplication defined in such a manner that the function $(\xi^2=4)=(3.\xi=12)$ has the value of the true for any argument whatever. Here an expression of logic may be used: "The concept *square-root of* 4 has the same extension as the concept *something of which three times its square is* 12." With those functions whose value is always a truth-value we can therefore say "extension of the concept" instead of "range of the function," and it seems suitable to say that a *concept* (*Begriff*) is a function of which the value is always a truth-value.

Hitherto I have only dealt with functions of a single argument, but we can easily pass over to *functions with two arguments*. Such functions are doubly in need of completion. A function with one argument is obtained when a completion by means of one argument has been effected. Only by means of a repeated completion do we arrive at an *object*, and this *object* is then called the *"value"* of the function for the pair of arguments. Just as the

letter "ξ" served with functions of one argument, I use here the letters "ξ" and "ζ" in order to indicate the twofold non-satisfaction of a function of two arguments, as, for example, in "$(\xi+\zeta)^2+\zeta$". By replacing "ζ" by "1", for example, we satisfy the function in such a way that we have in $(\xi+1)^2+1$ a function with only one argument. This manner in which we use the letters "ξ" and "ζ" must always be kept in mind when an expression like "the function $\Psi(\xi, \zeta)$" occurs.[20] I call the places in which "ξ" stands "ξ-argument-places", and those in which "ζ" stands "ζ-argument-places". I say that the ξ-argument-places are *"related"* (*verwandt*) to one another, and also the ζ-argument-places to one another, and I say that a ξ-argument-place is *not* related to a ζ-argument-place.

The functions with two arguments $\xi=\zeta$ and $\xi>\zeta$ have as value always a truth-value—at least if the signs "=" and ">" are defined in a suitable manner. I shall call such functions "relations". In the first relation, for example, 1 stands to 1, and in general every object to itself; in the second, for example, 2 stands to 1. I say that the object Γ "stands in the relation $\Psi(\xi, \zeta)$ to" the object Δ, if $\Phi(\Gamma, \Delta)$ is the true. I say that the object Δ *"falls under"* the concept $\Phi(\xi)$, if $\Phi(\Delta)$ is the true. It is presumed, of course, that both the functions $\Phi(\xi)$ and $\Psi(\xi,\zeta)$ have always truth-values as values.[21]

* * *

I have already said above that no assertion is to lie as

[20] Cf. note 18.

[21] Here there is a difficulty which may easily obscure the true position of things and thus rouse distrust of the correctness of my view. If we compare the expression "the truth-value of the circumstance that Δ falls under the concept $\Phi(\xi)$" with "$\Phi(\Delta)$", we see that to the "$\Phi(\Delta)$" properly corresponds "the truth-value of the circumstance that (Δ) falls under the concept $\Phi(\xi)$" and not "the concept $\Phi(\xi)$". The last words do not therefore really designate a concept (in my sense of the word), though they have the appearance of doing so in our linguistic form. With regard to the constrained position in which language here finds itself, cf. my essay "Ueber Begriff und Gegenstand" mentioned in note 14.

yet in a mere equation; by "$2+3=5$" only a truth-value is designated and it is not stated which of the two it is. Again, if I write "$(2+3=5)=(2=2)$" and presuppose that we know that $2=2$ is the true, yet I would not have asserted by that the sum of 2 and 3 is 5, but I would only have designated the truth-value of the circumstance that "$2+3=5$" denotes the same as "$2=2$". Thus we need a special sign to assert that something or other is true. For this purpose I write what I call a "sign of assertion" just before the name of the truth-value, so that if this sign is written just before "$2^2=4$",[22] it is asserted that the square of 2 is 4. I make a distinction between "*judgment*" (*Urtheil*) and "*thought*" (*Gedanken*), and understand by "judgment" the recognition of the truth of a "thought." I shall call the ideographic representation of a judgment by means of the sign of assertion an "ideographic theorem" or more shortly a "theorem." I regard this sign of assertion as composed of a vertical line, which I call "line of judgment" (*Urtheilsstrich*), and a short straight horizontal line proceeding from the middle of the vertical line and going toward the right, which I will simply call the "horizontal line" (*Wagerechte*). In my *Begriffsschrift* I called this last line the "line of content" (*Inhaltsstrich*) and at that time I expressed by the words "judicable content" (*beurtheilbarer Inhalt*) what I have now arrived at distinguishing into truth-value and thought.[23] The horizontal line most often occurs in combination with other signs, as it does here with the line of judgment, and is thus guarded against confusion with the minus sign. Wherever it occurs by itself it must be made somewhat longer than the minus sign for purposes of dis-

[22] I often use here the notations of sum, product, and power in order conveniently to form examples and to facilitate understanding by means of hints, although these signs are not yet defined in this place. But we must keep in view the fact that nothing is founded on the denotations of these signs.

[23] Cf. my essay "Ueber Sinn und Bedeutung" cited above.

tinction. I regard it as a name of a function in the way that "Δ" preceded by this sign denotes the true if Δ is the true, and the false if Δ is not the true. Of course the sign "Δ" must denote an *object*; names without denotation may not occur in our ideography. The above arrangement is made so that "Δ" preceded by a horizontal line denotes something under all circumstances if only "Δ" denotes something. If not, "ξ" preceded by a horizontal line would not denote a concept with sharp boundaries,—and thus would not denote a *concept* in my sense. I here use capital Greek letters as names denoting something without my saying what their denotations are. In the actual developments of my ideography they will not occur any more than "ξ" and "ζ". The above "ξ" preceded by a horizontal line denotes a function whose value is always a truth-value or, by what I have said, a concept. Under this concept falls the true and this only. Thus "$2^2=4$" preceded by a horizontal line denotes the same thing as "$2^2=4$", namely the true. In order to do away with brackets, I lay down that all which stands to the right of the horizontal line is to be regarded as a whole which stands at the argument-place of the function denoted by "ξ" preceded by a horizontal line, unless *brackets* forbid this. The sign "$2^2=5$" preceded by a horizontal line denotes the false and thus the same as "$2^2=5$", whereas "2" preceded by a horizontal line denotes the false, and thus something different from the number 2. If "Δ" is a truth-value, Δ preceded by a horizontal line is the same truth-value, and thus the equation of "Δ" to "Δ" preceded by a horizontal line denotes the true. But this equation denotes the false is Δ is not a truth-value; so that we can say that it denotes the truth-value of the circumstances that Δ is a truth-value.

Thus the function "$\Phi(\xi)$" preceded by a horizontal line, denotes a concept and the function "$\Psi(\xi,\zeta)$" pre-

ceded by a horizontal line, denotes a relation, whether or not $\Phi(\xi)$ is a concept and $\Psi(\xi, \zeta)$ is a relation.

Of the two signs out of which the sign of assertion is composed the line of judgment alone contains the assertion.

We need no sign to declare that a truth-value is the false, if only we have a sign by which either truth-value is changed into the other. This sign is also indispensable on other grounds. I now lay down that the value of the function denoted by "ξ" preceded by a horizontal line from the middle of which hangs a small vertical line directed downward and called the "line of denial" (*Verneinungs-strich*), so that the whole is like a sign of assertion turned round on its face, is to denote the false for every argument for which the value of the function denoted by "ξ" preceded by a horizontal line is the true. For all other arguments the function under definition is to be the true. The function thus defined may be called "the negation of ξ", and thus its value is always a truth-value; it is a concept under which all objects fall with the single exception of the true. From this it follows that horizontal lines, whether or not they form part of a sign of negation, can be combined with immediately preceding or following simple horizontal lines in such a way that the latter, so to speak, lose their separate existence and *melt into* the former (*Verschmelzung der Wagerechten*).

Thus "the negation of $2^2 = 5$" denotes the true; and thus we may put the sign of assertion so as to join on to the left of the sign of negation. We may assert, too, the negation of 2.

I have already used the sign of equality to form examples, but it is necessary to lay down something more accurate about it. The sign "$\Gamma = \Delta$" is to denote the true if Γ is the same as Δ, and the false in all other cases.

In order to dispense with brackets as far as possible, I lay down that all which stands on the left of the sign

of equality as far as the nearest horizontal line is to denote the ξ-argument of the function $\xi = \zeta$, in so far as *brackets* do not forbid this; and that all which stands on the right of the sign of equality as far as the next sign of equality is to denote the ζ-argument of that function in so far as *brackets* do not forbid this.

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A CHINESE POET'S CONTEMPLATION OF LIFE.

INTRODUCTION.

MY attention has repeatedly been called to the poetry of Su Tung P'o (also briefly named "Su Hsi"), especially to his thoughtful meditation on an excursion by boat to the Scarlet Cliff. In this poem he comments on the transiency of life, and referring to the law of change as represented by the phases of the moon he finds the underlying permanence symbolized by the river which remains the same although its waters pass on without a halt.

The original was kindly furnished me by Mr. Sawland J. Shu, president of the Technological College at Nanking, while a literal translation was procured through Prof. Frederick G. Henke from Mr. W. T. Tao and another one from Prof. King Shu Liu, of the University of Nanking. Professor Henke further informed me on the authority of Prof. William F. Hummel that a prose translation by Prof. Herbert A. Giles was published in the *University of Nanking Magazine* and republished together with other Chinese poems collected in the volume entitled *Gems of Chinese Literature.*

Professor Giles says that Su Tung P'o was "even a greater favorite with the Chinese literary public" than the famous Ou-Yang Hsiu.[1] So we may regard Su Tung P'o as easily a genius of first rank. Professor Giles says of him:

"Under his hands, the language of which China is so proud may be said to have reached perfection of finish, of art concealed. In subtlety of reasonings, in the lucid expression of abstractions, such as in English too often elude the faculty of the tongue, Su Tung P'o is an unrivalled master."

Even a rough translation of his poems will impress the reader

[1] Ou-Yang Hsiu lived 1017-1072 A.D. Professor Giles says of him: "A leading statesman, historian, poet, and essayist of the Sung dynasty. His tablet is to be found in the Confucian temple, an honor reserved for those alone who have contributed to the elucidation or dissemination of Confucian truth."

with the versatility as well as the profundity of his poetic flights, and here I venture to present his famous poem on "The Scarlet Cliff" in English blank verse which seems to be the appropriate form for this kind of thought. I hope that it will be a fair example of Chinese literature in its noblest accomplishment.

There are some people who have little appreciation of the beauties of Chinese literature and have nothing but ridicule or even contempt for it. With reference to one of these haughty scoffers Professor Giles adds with grim humor:

"On behalf of his (Su Tung P'o's) honored manes I desire to note my protest against the words of Mr. Baber, recently spoken at a meeting of the Royal Geographical Society, and stating that 'the Chinese language is incompetent to express the subtleties of theological reasoning, just as it is inadequate to represent the nomenclature of European science.' I am not aware that the nomenclature of European science can be adequately represented even in the English language; at any rate, there can be no comparison between the expression of terms and of ideas, and I take it the doctrine of the Trinity itself is not more difficult of comprehension than the theory of 'self-abstraction beyond the limits of an external world,' so closely reasoned out by Chuang Szu. If Mr. Baber merely means that the gentlemen entrusted with the task have proved themselves so far quite incompetent to express in Chinese the subtleties of theological reasoning, then I am with him to the death."

Mr. K. S. Liu sends with his translation these further remarks concerning Su Hsi, the classical philosopher of Chinese *belles lettres*:

"This poem was composed by Su Hsi, a famous Chinese poet who flourished 1036-1101. Owing to the intrigues of his political enemies he was exiled to Hwang-Cheo, a place in the province of Hu Peh. While there he made a visit to a place called Chi Pi (literally Red Wall), made famous by the battle which took place there between Tsao-tsao and Cheo-yu (two historical characters in the period of the Three Kingdoms). The poem is an account of this visit and a description of the feelings it aroused in him. Like many other poets who consider poetry an embodiment in symbols of one's inner spiritual experiences, he shows in the poem, first, the ephemeral nature of human existence with all its paraphernalia, and then how in the contemplation of nature one can transcend the mutations of time and be one with the eternal order. In this state one can rise above the vicissitudes of life."

The poem begins by giving the date of Su Hsi's excursion to the Scarlet Cliff. The year reads in Chinese characters *jan süh,* and we here encounter the difficulty of reproducing the Chinese

前游赤壁賦　　蘇軾
壬戌之秋七月既望蘇子與客泛舟
游於赤壁之下清風徐來水波不興
舉酒屬客誦明月之詩歌窈窕之章
少焉月出於東山之上徘徊於斗牛
之間白露橫江水光接天縱一葦之
所如凌萬頃之茫然浩浩乎如馮虛
御風而不知其所止飄飄乎如遺世
獨立羽化而登仙於是飲酒樂甚扣
舷而歌之歌曰桂棹兮蘭槳擊空明
兮泝流光渺渺兮予懷望美人兮天
一方客有吹洞簫者倚歌而和之其
聲嗚嗚然如怨如慕如泣如訴餘音
裊裊不絕如縷舞幽壑之潛蛟泣孤
舟之嫠婦蘇子愀然正襟危坐而問
客曰何為其然也客曰月明星稀烏

CHINESE TEXT.

method of determining chronology. For this they make use of the sexagenary cycle by repeating five times the twelve branches and six times the ten stems (see the author's *Chinese Thought,* p. 4).

The meaning of *jan* (pronounced *zhan*) is the "germ in the womb," and it "denotes the ninth of the ten stems; it is connected with the north and running water." It means "great, full" and also

CHINESE TEXT.

17 鵲南飛此非曹孟德之詩乎東望夏
18 口西望武昌山川相繆鬱乎蒼蒼此
19 非孟德之困於周郎者乎方其破荊州
20 下江陵順流而東也舳艫千里旌旗蔽
21 空釃酒臨江橫槊賦詩固一世之雄也而
22 今安在哉況吾與子漁樵於江渚之上侶
23 魚蝦而友麋鹿駕一葉之扁舟舉匏樽
24 以相屬寄蜉蝣於天地渺滄海之一粟
25 哀吾生之須臾羨長江之無窮挾飛
26 仙以遨遊抱明月而長終知不可乎
27 驟得託遺響於悲風蘇子曰客亦知夫
28 水與月乎逝者如斯而未嘗往也盈虛
29 者如彼而卒莫消長也蓋將自其變者
30 而觀之則天地曾不能以一瞬自其不變
31 者而觀之則物與我皆無盡也而又何羨
32 乎且夫天地之間物各有主苟非吾之所

"to flatter and adulate." As the ninth of the ten stems it denotes swollen water, hence we translate it "billow." The other character *süh* which is the eleventh of the twelve branches denotes in its

horary significance the hour 7-9 P. M., called the "dog hour." We here translate it by "hound." To Chinamen this denotation of the year is very familiar, but it is difficult to reproduce its exact significance in a poetic translation in English. The "billow hound" year corresponds in our chronology to 1082 A. D., which is the fifty-eighth year in the sexagenary cycle under the Sung dynasty. The latter being a matter of course in the poet's day is not mentioned in the Chinese text.

33 有雖一毫而莫取唯江上之清風山間
34 之明月耳得之而為聲目遇之而成色
35 取之不盡用之不竭是造物者之無盡
36 藏也而吾與子之所共適客喜而笑洗
37 盞更酌肴核既盡杯盤狼藉相與枕
38 藉乎舟中不知東方之既白
39 千九百十四年六月宋王仕鴻書

CHINESE TEXT.

The songs "To the Bright Moon" and "To the Modest Maid" mentioned in the poem are probably the odes known as I, XII, 8 and I, III, 17 of the *Shih King*, the canonical collection of ancient Chinese songs. In the translation of William Jennings (*The Shi King*, pages 151 and 69) they read as follows:

To the Bright Moon.

O Moon that climb'st effulgent!
O ladylove most sweet!

Would that my ardor found thee more indulgent!
  Poor heart, how dost thou vainly beat!

O Moon that climb'st in splendor!
  O ladylove most fair!
Couldst thou relief to my fond yearning render!
  Poor heart, what chafing must thou bear!

O Moon that climb'st serenely!
  O ladylove most bright!
Couldst thou relax the chain I feel so keenly!
  Poor heart, how sorry is thy plight!

To the Modest Maid.

A modest maiden, passing fair to see,
Waits at the corner of the wall for me.
I love her, yet I have no interview:—
I scratch my head—I know not what to do.

The modest maid—how winsome was she then,
The day she gave me her vermilion pen!
Vermilion pen was never yet so bright—
The maid's own loveliness is my delight.

Now from the pasture lands she sends a shoot
Of couchgrass fair; and rare it is, to boot.
Yet thou, my plant (when beauties I compare),
Art but the fair one's gift, and not the Fair!

There is some doubt, according to Professor Giles, whether the Scarlet Cliff visited by Su Hsi was really the place of battle as the latter assumes, but the poem remains of the same significance even if Su Hsi was mistaken, and we need feel no concern about it.

P. C.

THE SCARLET CLIFF.

It was the Billow-Hound year of House Sung:
The seventh moon was on the wane, when I
Was down stream drifting in a boat with friends
On an excursion to the Scarlet Cliff.

The evening breeze so gently blew that scarce
The water rippled on its smooth expanse.
I filled the cups and bade my friends to sing
The ode "To the Bright Moon," and then they chanted
The lay melodious "To the Modest Maid."

Slowly the moon rose o'er the eastern hills,
And passed between the Wain and Capricorn,
Shedding her silver beams upon the water,
To link our world below with heaven above.

In such surroundings, infinite in charm,
Our skiff was freely gliding,—traveling
Unchecked through space, unmindful whither bound;
Like gods we moved in a transcendent realm:
I poured out a libation for our joy,
And beating time on our boat's wooden rim,
I sang these verses in sad exaltation:

"Our olive boat with orchid oars propelled,
Breaks splashing through the moonlit glittering wave;
In lovelorn loneliness I here am held,
From friends who now lie buried in the grave."

One of my guests accompanied the song
Upon his flageolet, with proper notes
To suit the music to the sentiment
Of plaintive moods, in sounds that wove unbroken
Their silken threads around our company.
The music stirred the dragon in the deep
And moved the the boatswain's widow unto tears.
"And why is that?" I asked in pensive query
My cherished guest. "Why does thy magic art
So powerfully affect us all?" Said he:

"Few stars are seen and yet the moon shines bright,
To southern lands the raven wings his flight.

"Was this not uttered here by Tsao Meng Te,
Here, eastward of Hsia-K'ou, west of Wu-Chang,
Where hill and stream in wild luxuriance blend?
'T is here Meng Te was routed by Chou Yü.
Before him lay Ching-chou. Kiang-ling he conquered,
And eastward did he push upon the river;
His warships, prow to stern, stretched thousand miles,
The banners of his troops darkened the sun.
Then a libation he poured out, and nearing
The Scarlet Cliff, the hero of his age,
On horseback, clad in armor, spake those words!
Yet where is he to-day? And what are we?
To-day we fish and gather fuel here
On river isles where shrimps are our companions
And deer our friends. We paddle here about
In frail canoe and drink companionship
From flasks of gourd. How transient is the life
Of creatures as ephemeral as we.
Tossed o'er the ocean like a husk of straw,
We are mere twinklings on the river Time;
Oh, could I be the stream itself which rolls
Incessantly and without end! Alas!
Could I but clasp the bright and beauteous moon
Close to my heart and dwell with her in heaven!
Yet unfulfilled remain my deep-felt yearnings
Which find expression in melodious strain."

"But you my friend," replied I questioning,
"Do you well comprehend the mystery
Of this great river and the changing moon?
Past flows the water but 'tis never gone;
The moon is waning, but again 'twill wax.

So I with this great world, all in a change—
E'en Heaven and Earth are transient constantly—
Myself, and also thou, in this same sense
Viewed as a whole, live on eternally.
Why then lament? Thou long'st for what thou hast!"
And further musing on life's complex problems
Continued I: "Whate'er our senses hold
Is owned by him who feels it, who enjoys it.
For nothing can I take unless I own it,
The bracing breeze, the landscape of the river,
The moon above the valleys, gorgeous sights
Enrapturing the eye, and all the sounds
Which greet the ear, all are enjoyed by me.
All these are mine, and without let or hindrance
Are they the gifts of God, unstintedly
Given to man—indeed to all mankind.
And we enjoy them now."
He smiled approval—
My friend; he threw away the dregs of wine
And had his cup refilled up to the brim.

Thus finishing our feast we laid us down
To rest among the scattered cups and plates,
While in the distant east dim streaks of light
Appeared as heralds of another day.

CRITICISMS AND DISCUSSIONS.

LEIBNIZ AND LOCKE.

John Locke, the founder of the sensationalist school, who formulated the principle of his philosophy in the statement *Nihil est in intellectu quod non antea fuerit in sensu,* and who therefore on the one hand denied innate ideas and on the other claimed that all knowledge rises from experience, devotes to an investigation of truth Chapter V, and also part of Chapter VI of his famous work *On the Human Understanding* from which we make the following extracts:

"'What is truth?' was an inquiry many ages since; and it being that which all mankind either do or pretend to search after, it cannot but be worth our while carefully to examine wherein it consists; and so acquaint ourselves with the nature of it, as to observe how the mind distinguishes it from falsehood.

"Truth then seems to me, in the proper import of the word, to signify nothing but the joining or separating of signs, as the things signified by them do agree or disagree one with another. The joining or separating of signs here meant, is what by another name we call 'proposition.' So that truth properly belongs only to propositions: whereof there are two sorts, viz., mental and verbal; as there are two sorts of signs commonly made use of, viz., ideas and words...

"We must, I say, observe two sorts of propositions that we are capable of making:

"First, Mental, wherein the ideas in our understandings are, without the use of words, put together or separated by the mind perceiving or judging of their agreement or disagreement.

"Secondly, Verbal propositions, which are words, the signs of our ideas, put together or separated in affirmative or negative sentences. By which way of affirming or denying, these signs, made by sounds, are, as it were, put together or separated one from an-

other. So that proposition consists in joining or separating these signs, according as the things which they stand for agree or disagree....

"When ideas are so put together or separated in the mind, as they or the things they stand for do agree or not, that is, as I may call it 'mental truth.' But truth of words is something more, and that is the affirming or denying of words of another, as the ideas they stand for agree or disagree: and this again is twofold; either purely verbal and trifling or real and instructive, which is the object of real knowledge....

"Though our words signify nothing but our ideas, yet being designed by them to signify things, the truth they contain, when put into propositions, will be only verbal when they stand for ideas in the mind have not an agreement with the reality of things. And therefore truth, as well as knowledge, may well come under the distinction 'verbal' and 'real'; that being only verbal truth wherein terms are joined according to the agreement or disagreement of the ideas they stand for, without regarding whether our ideas are such as really have or are capable of having an existence in nature. But then it is they contain real truth when these signs are joined as our ideas agree; and when our ideas are such as we know are capable of having an existence in nature: which in substances we cannot know but by knowing that such have existed.

"Truth is the marking down in words the agreement or disagreement of ideas as it is. Falsehood is the marking down in words the agreement or disagreement of ideas otherwise than it is. And so far as these ideas thus marked by sounds agree to their archetypes, so far only is the truth real. The knowledge of this truth consists in knowing what ideas the words stand for, and the perception of the agreement or disagreement of those ideas, according as it is marked by those words....

"Certainty is twofold; certainty of truth, and certainty of knowledge. Certainty of truth is, when words are so put together in propositions as exactly to express the agreement or disagreement of the ideas they stand for, as really it is. Certainty of knowledge is, to perceive the agreement or disagreement of ideas, as expressed in any proposition. This we usually call 'knowing,' or 'being certain of the truth of any proposition.'"

His great critic Leibniz wrote a voluminous book[1] to refute

[1] *New Essays Concerning Human Understanding.* Translated by A. G. Langley. 2d ed., Chicago and London, 1916.

Locke's sensationalism, pointing out that what Locke called reflection was not a product of sensation. He amended Locke's principle to read: *Nihil est in intellectu quod non antea fuerit in sensu, nisi intellectus ipse*, and this amendment upset Locke's very lucid but superficial arguments. According to Leibniz the senses furnish us the material for positive knowledge but they offer nothing but particular instances, not methods, nor principles, nor general truths. Brutes have the same sensations as man, but brutes can never attain to necessary propositions. These conceptions of necessary propositions are innate in the human mind. The human mind is not a *tabula rasa,* but contains certain principles which, in the measure that experience furnishes the occasion, develop into ideas of eternal and necessary verities.

From this standpoint Leibniz distinguishes two kinds of truths, necessary truths and contingent truths; the former are the eternal verities as instanced by mathematics, the latter the knowledge of particular facts furnished by experience. God is the ultimate source of both kinds of truth; the eternal verities correspond to his intellect, the contingent truths to his will. The former are such and can not be different because God is such; the latter could be different but are not because God willed them to be as they are and not otherwise. Necessary truths reveal to us what is possible and what impossible. Thus, e. g., a regular decahedron (i. e., a figure bounded by ten equal plane surfaces) is impossible, and "all intelligible ideas have their archetype in the eternal possibilities of things."

In reply to Locke's view of certainty, Leibniz says:

"Our certitude would be small, or rather nothing, if it had no other basis of simple ideas than that which comes from the senses. Have you forgotten, sir, how I have shown that ideas are originally in our mind, and that indeed our thoughts come to us from the depths of our own nature, other creatures being unable to have an immediate influence upon the soul? Besides, the ground of our certitude in regard to universal and eternal truths is in the ideas themselves, independently of the senses, just as ideas pure and intelligible do not depend on the senses, for example, those of being, unity, identity, etc. But the ideas of sensible qualities, as color, savor, etc., (which in reality are only phantasms) come to us from the senses, i. e., from our confused perceptions. And the basis of the truth of contingent and particular things is in the succession

which causes these phenomena of the senses to be rightly united as the intelligible truths demand."

It is not our intention to criticize any one of the philosophers but we wish to point out how far and in what respect we agree with Leibniz's views as here outlined. We select Leibniz because his philosophy is less onesided than any other and has incorporated all considerations, religious, scientific, mathematical and historical. What he calls innate ideas reflecting the eternal and necessary truths whose source lies in God, we denote as the purely formal and we have shown that purely formal conceptions have been gained by abstraction. Man alone has the faculty of abstraction and so he alone is capable of producing and operating with purely formal conceptions such as numbers, geometrical figures, the notion of mathematical or pure space, logical syllogisms, the formulas of causation and of the conservation of substance and energy. The principle pervading the function of these concepts is called reason, and reason truly reflects the cosmic order, which is due to the efficiency of purely formal interrelations—the so-called purely formal laws. Our senses furnish us particulars only, and these particulars, which are innumerable isolated sense-impressions, would remain a chaos of disconnected items if they were not classified and systematized according to purely formal laws. The point overlooked by Leibniz and also later on by Kant is the question as to the origin of mind. The framework of reason, man's logical faculty, his notion of numbers and of space relations have indeed originated through experience as Locke claimed, but it was experience in a wider sense than either Locke or Leibniz conceived it to be. Experience in those days meant sense-experience, or the purely sensory element of sentient creatures. In this sense Leibniz is right that no amount of sense-impressions can bring forth an eternal or universal or necessary idea. Locke on the other hand, conscious of the fact that man was in possession of universal and necessary concepts and admitting no other source of knowledge than experience, insisted on the proposition that all ideas, even the most complicated ones, were derived from sensations, as which he understands experience to be.

Now it is obvious that there is nothing purely sensory, Sensations are possessed of forms and the formal impresses itself together with sense impressions upon sentient creatures. We have on

other occasions set forth how sensory impressions are by a mechanical necessity so grouped that they are registered together, the particular ones being subsumed under the more general so that all of them build up a well-arranged system constituting a logical framework of types. This framework is the mind which is built up not of mere sensations, but of the interrelations of sense-impressions according to their various forms. Experience in the current sense includes the form of the sensory, and in this sense the faculty of conceiving purely formal relations has indeed arisen from experience.

The sensationalist school identifies the sense element of our knowledge with the formal and overlooks their radical difference. We must insist against the sensationalist school that everything formal is radically different from the sensory. The sensory is always particular while the formal can be generalized. By leaving out of sight everything particular our thought can operate in a field of pure relations, and we can exhaust all their possibilities. We can say what is possible as well as what is impossible and (all interference of unexpected particulars being excluded) we can also say what result will always be obtained under definite given conditions. We can exhaust all possibilities of the purely formal and can systematize the whole field. What will always be, is called "necessary," and so these propositions which are inevitable are called by Leibniz "eternal truths."

We agree with Leibniz that the source of these eternal truths is God; nay we go one step further in definiteness and claim that the eternal verities, of which our human notions of eternal truths are mental reflections, are God himself. All depends on our definition of God. Together with the whole cosmic order the necessary truths constitute an eternal omnipresence, an efficient system of norms which mould the world and determine all things. They form a kind of spiritual, or purely formal organism, a superpersonal presence which is the ultimate *raison d'être* and determinant of all things, the cosmos in its entirety as well as all particular events that happen in the course of its being.

Any one who has once grasped the deep significance of the purely formal will have liberated his mind forever of the superstitious, mystical or allegorical conceptions of the deity, but he will at the same time understand the truth that underlies the God-idea and thus he will know the real nature of the true God, whose exist-

ence is not a matter of belief, but a scientific certainty. All former proofs of the existence of God were necessarily failures, because in all cases the attempt was made to prove the existence of an anthropomorphic God with arguments that prove the true God, the eternal norm of being, and here the argument breaks down, because it no longer applies to the idea of an anthropomorphic God.

Leibniz has not overcome the mystical conception both of God and truth. He has unfortunately adopted the very primitive conception of an atomic nature of reality which is described in his monadology. It is strange that a man of his caliber did not see how contradictory is the idea of God as the central monad. On the other hand his theory is vindicated if we interpret his God to be the universal and omnipresent norm that regulates every event and constitutes the cosmic order of the world.

Insisting on the unity of the soul, Leibniz conceived all unities as local units, and these innumerable local units, the monads, were conceived as centers of force endowed with feeling and an entelechy, which means that they were capable of pursuing purposes. At the same time Leibniz held them to be separate entities, so as to render their cohesion and interaction a profound problem which could be solved only by the bold hypothesis of the preestablished harmony.

The problem of unity together with all problems of combination and configuration belongs in the domain of pure form. Combination of several parts working in cooperation constitute a unity and introduce something new. It did not exist before and will break to pieces again, but the law of its combination remains forever and constitutes the eternal background of its existence. The sensationalist school misses the main point of all philosophical considerations and thus loses the essence of the significance of religion; but Leibniz who discovers the weak spot in their arguments has not succeeded in persenting a satisfactory solution of the problem but ends in proclaiming a mystical God-conception and a dogmatic proclamation of a preestablished harmony. P. C.

EXISTENTS AND ENTITIES.[1]

That we must distinguish between what we may call "having existence" and "having entity or being" becomes evident when we look somewhat closely at ordinary mathematical propositions. A class (or system, or aggregate) M is said to "exist" when it has

[1] Cf. *Monist,* Jan. 1910, Vol. XX, p. 114, note 85.

at least one member;[2] whereas, when mathematicians speak of, for example, "the existence of roots of an equation" or "the existence of the definite integral of a continuous function," they use the word "existence" in another sense: the roots or the integral are not classes, but individuals constructed out of mathematical concepts to supply an answer to certain questions. We can, of course, consider such an individual as *the* member of the class (N) whose sole member is this individual, and can then consider the second kind of mathematicians' existence-proofs as proofs of the existence of the class N; but we should, for the sake of clearness, avoid speaking of the "existence" of the member[3] of N, and use some such word as "entity" or "being" instead.

Mr. B. Russell[4] has thus distinguished *being* and *existence* in 1901: "*Being* is that which belongs to every conceivable term, to every possible object of thought—in short to everything that can possibly occur in any proposition, true or false, and to all such propositions themselves. Being belongs to whatever can be counted. If A be any term that can be counted as one, it is plain that A is something, and therefore that A is. 'A is not' must always be either false or meaningless. For if A were nothing, it could not be said not to be; 'A is not' implies that there is a term A whose being is denied, and hence that A is. Thus unless 'A is not' be an empty sound, it must be false—whatever A may be, it certainly is. Numbers, the Homeric gods, relations, chimeras, and four-dimensional spaces all have being, for if they were not entities of a kind, we could make no propositions about them. Thus being is a general attribute of everything, and to mention anything is to show that it is.

"*Existence*, on the contrary, is the prerogative of some only amongst beings. To exist is to have a specific relation to existence —a relation, by the way, which existence itself does not have. This shows, incidentally, the weakness of the existential theory of judgment—the theory, that is, that every proposition is concerned with something that exists. For if this theory were true, it would still be true that existence itself is an entity, and it must be admitted that existence does not exist. Thus the consideration of existence itself

[2] Cf., e. g., Dedekind, *Was sind und was sollen die Zahlen?* 2d ed., Braunschweig, 1893, pp. 5, 12; or *Essays on the Theory of Numbers*, Chicago, 1901, pp. 49, 58; Russell, *The Principles of Mathematics*, Cambridge, 1903, pp. 21, 32.

[3] Of course, the member of N may be itself a class and may thus "exist," but we obviously need not consider this further.

[4] *Mind*, N. S., Vol. X, No. 39, 1901, pp. 310-311.

leads to non-existential propositions, and so contradicts the theory...."

This doctrine was repeated in Mr. Russell's *Principles of Mathematics*;[5] the existence-theorems of mathematics were said[6] to be "proofs that the various classes defined are not null," and the earlier statement[7] that these theorems are proofs "that there are entities of the kind in question" must not be taken to mean what it apparently expresses.

While Mr. Russell emphasized the distinction between *entity* and *existence*, it does not seem that at that time he quite realized the full bearings of the question, at least in mathematics. He attributed a denotation to every term that can possibly occur in a proposition. Thus "the round square" had a denotation, and the only further existence-question in logic and mathematics was whether the numbers—at least such as were defined as classes—, classes of spaces, and so on, could be proved to "exist,"—whether members of the classes in question could be constructed by logical methods provided that the initial postulates are granted.

* * *

Before going on to discuss the clear separation of the important question of entity from the less important question of existence, which came in Mr. Russell's later works, we will refer to the very strong tendency, even among logicians and mathematicians, to attribute a denotation to every denoting phrase.

Thus, H. MacColl[8] remarked that a symbol which corresponds to nothing in our universe of admitted realities, has, nevertheless, "like everything else named," a *symbolical* entity. In his sixth paper on "Symbolic Reasoning,"[9] MacColl attempted to give a simple theory of the existential import of propositions.

By $e_1, e_2, e_3, \ldots$, he denoted "our universe of *real existences,*" and by $o_1, o_2, o_3, \ldots$, "our universe of *non-existences,* that is to say, of *unrealities,* such as *centaurs, nectar, ambrosia, fairies,* with self-contradictions, such as *round squares, square circles, flat spheres,*

[5] Pp. 449-450; cf. pp. 43, 71.

[6] *Ibid.*, p. 497.

[7] *Ibid.*, p. vii.

[8] *Symbolic Logic and its Applications*, London, 1906, p. 42; MacColl here and elsewhere used the word "existence" where we use "entity." Cf. *Mind*, N. S., Vol. XI, 1902, pp. 356-357.

[9] *Mind*, N. S., Vol. XIV, 1905, pp. 74-81; cf. *Symbolic Logic and its Applications*, pp. 5, 76-78.

etc."; the "symbolic universe, or universe of discourse," S, may consist either wholly of realities, wholly of unrealities, or partly of realities and partly of unrealities.... If A denotes an individual or a class, any intelligible statement $\phi(A)$ containing the symbol A, implies that the individual or class represented by A has a *symbolic* existence; but whether the statement $\phi(A)$ implies that that which A denotes has a *real* or *unreal* or (if a class) partly real and partly unreal existence, depends upon the context."

We will pass over the discussion between Messrs. MacColl and A. T. Shearman[10] on the interpretation of the Boolian equation "O=OA," and come to Mr. Russell's articles of 1905,[11] in which the theory of non-entity was, it seems, for the first time treated satisfactorily.

The sense in which the word "existence" is used in symbolic logic is a definable and purely technical sense. To say that A exists means that A is a class which has at least one member. Thus whatever is not a class does not exist in this sense; and among classes there is just one that does not exist, namely, the null-class. MacColl's two universes of existences and non-existences are not to be distinguished in symbolic logic, and each of them is identical with the null-class. There are no centaurs; "x is a centaur" is false whatever value we give to x, even when we include values which do not "exist" in the meaning which occurs in philosophy and daily life, such as numbers or propositions.

"The case of nectar and ambrosia is more difficult, since these seem to be individuals, not classes. But here we must presuppose definitions of nectar and ambrosia: they are substances having such and such properties, which, as a matter of fact, no substances do have. We have thus merely a defining concept for each, without any entity to which the concept applies. In this case, the concept is an entity, but it does not denote anything....These words [such as nectar and ambrosia] have a *meaning,* which can be found by looking them up in a classical dictionary, but they have not a *denotation*: there is no entity, real or imaginary, which they point out."

[10] *Mind,* N. S., Vol. XIV, 1905, pp. 78-79, 295-296, 440, 578-580; Vol. XV, 1906, pp. 143-144; and Shearman's book *The Development of Symbolic Logic; a Critical-Historical Study of the Logical Calculus,* London, 1906, pp. 161-171.

[11] "The Existential Import of Propositions," *Mind,* N. S., Vol. XIV, 1905, pp. 398-401; "On Denoting," *ibid.,* pp. 479-493.

The last sentence refers to Frege's[12] distinction of *Sinn* (meaning) and *Bedeutung* (denotation).

A point of passing interest in connection with an attempt at the solution of a mathematical paradox, referred to later, is this sentence in MacColl's reply:[13] "I may mention, as a fact not wholly irrelevant, that it was in the actual application of my symbolic system to concrete problems that I found it absolutely necessary to label realities and unrealities by special symbols e and o, and to break up the latter class into separate individuals, o_1, o_2, o_3, etc., just as I break up the former into separate individuals e_1, e_2, e_3, etc."

When a phrase which in form is denoting, and yet does not denote anything,—e. g., "the present king of France,"—occurs in the statement of a proposition, the question as to the interpretation of propositions in whose verbal expression this phrase occurs arises, and Mr. Russell, in the article "On Denoting" referred to, succeeded in assigning a meaning to every proposition in whose verbal expression any denoting phrases—whether they appear to denote something or nothing at all, e. g., everything, nothing, something, a man, every man, no man, the father of Charles II, the present king of France—occur. It is not necessary to assume that denoting phrases ever have any meaning in themselves.

The theory of MacColl and the allied theory of Meinong were rejected by Mr. Russell[14] because they conflict with the law of contradiction. If any grammatically correct denoting phrase stands for an *object* although such objects may not *subsist*, such objects are apt to infringe the law of contradiction. Thus it is contended that the round square is round, and also not round.

To solve the paradoxes that appear in the mathematical theory of aggregates, Mr. Russell treated classes and relations in the same way as he treated denoting phrases.[15]

Poincaré, among others, recognized that all the paradoxes of the modern theory of aggregates, such as those of Burali-Forti, Russell and Richard, arise from a kind of vicious circle which may be expressed, in the language of Peano, thus: Everything which

[12] "Ueber Sinn und Bedeutung," *Zeitschr. für Phil. und phil. Kritik*, Vol. C, 1892, pp. 25-50.

[13] *Mind*, N. S., Vol. XIV, 1905, p. 401.

[14] *Ibid.*, pp. 491, 482-483.

[15] "On Some Difficulties in the Theory of Transfinite Numbers and Order Types," *Proc. Lond. Math. Soc.* (2), Vol. IV, 1906, pp. 29-53 (cf. especially the part on the "No-Classes Theory"); "Les Paradoxes de la Logique," *Rev. de Métaphys. et de Morale*, Vol. XIV, 1906, pp. 627-650.

contains an apparent variable must not be one of the possible values of this variable.[16] But Poincaré did not perceive that if we wish to avoid such vicious circles we must have recourse to a fundamental re-moulding of logical principles, more or less analogous to the "no classes" theory. To have shown this seems to be one of Mr. Russell's greatest merits; simply because practically all the other mathematicians who have interested themselves in the paradoxes did not realize this important fact. Thus, said Mr. Russell,[17] the method by which Poincaré tried to avoid the vicious circle consists in saying that when we assert that "all propositions are true or false," which is the law of the excluded middle, we exclude tacitly the law of the excluded middle itself. The difficulty is to make this tacit exclusion legitimate without falling into the vicious circle. If we say, "All propositions are true or false, excepting the proposition that every proposition is true or false," we do not avoid the vicious circle. For this is a judgment bearing on all propositions, viz.: "All propositions are either true or false. or identical with the proposition that all propositions are true or false." And that supposes that we know the meaning of "all propositions are true or false," where *all* has no exception. That comes to defining the law of the excluded middle by: "All propositions with the exception of the law of the excluded middle are true or false," where the vicious circle is flagrant. We must, then, find a means to formulate the law of the excluded middle in such a way that it does not apply to itself.

On the details of the new construction of logic in such a way that the paradoxes are avoided while nearly all of the work of Cantor on the transfinite is preserved, we must refer to Mr. Russell's works of 1908 and 1910.[18] Mr. Russell's method of avoiding the paradoxes in question is by what he called the "theory of types," and the object of this theory was shortly described by Dr. Whitehead and Mr. Russell[19] as follows: "The vicious circles in question arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a

[16] We may also express this principle as follows: A collection of objects may not contain members which can only be defined by means of the collection as a whole.

[17] *Rev. de Métaphys. et de Morale,* Vol. XIV, pp. 644-645.

[18] "Mathematical Logic as Based on the Theory of Types," *Amer. Journ. of Math.,* Vol. XXX, 1908, pp. 222-262; A. N. Whitehead and B. Russell, *Principia Mathematica,* Vol. I, Cambridge, 1910, pp. 39-88.

[19] *Op. cit.,* p. 39.

whole. Thus, for example, the collection of *propositions* will be supposed to contain a proposition stating that 'all propositions are either true or false.' It would seem, however, that such a statement could not be legitimate unless 'all propositions' referred to some already definite collection, which it cannot do if new propositions are created by statements about 'all propositions.' We shall, therefore, have to say that statements about 'all propositions' are meaningless. More generally, given any set of objects such that, if we suppose the set to have a total, then such a set cannot have a total. By saying that a set has 'no total,' we mean, primarily, that no significant statement can be made about 'all its members.' Propositions, as the above illustration shows, must be a set having no total. The same is true, as we shall shortly see, of propositional functions, even when these are restricted to such as can significantly have as argument a given object *a*. In such cases, it is necessary to break up our set into smaller sets, each of which is capable of a total. This is what the theory of types aims at effecting."[20]

* * *

In the next place, we shall go back four or five years in time, and see how the distinction between *entity* and *existence* became necessary in a mathematical investigation which is somewhat familiar to me. If I consider, at rather greater length than it deserves, my own work of 1903 and 1904[21] on the contradiction of Burali-Forti and its bearings on the theory of well-ordered aggregates, it is merely because familiarity with this investigation enables me to point out a small, unobserved merit which it has, in distinguishing *entity* from *existence,* and also to give yet another illustration of the tendency—which seems particularly common with mathematicians—of holding to the belief in the being or existence or subsistence in some sense, of a non-entity.

Burali-Forti had found, in 1897, the now well-known contradiction arising from the fact that 'the ordinal type of the whole series of (finite and transfinite) ordinal numbers' appears both to be and not to be the greatest ordinal number. From this I concluded, in 1903, that there are no such things as "the type" and

[20] The theory of logical types was described, in ordinary language, in *op. cit.*, pp. 39-68; and the theory of denoting was explained in the chapter on "Incomplete Symbols" (*ibid.*, pp 69-88).

[21] A general account of these investigations is contained in my paper, written in Peano's international (uninflected) Latin: "De Infinito in Mathematica," in ***Revista de Mathematica,*** Vol. VIII.

"the cardinal number" of the series just referred to. Hence, by a tacit use of an axiom afterwards stated explicitly by Zermelo, I concluded that every aggregate which has a cardinal number and every series which has a type can be well-ordered. The use of Zermelo's axiom was, with me as with most mathematicians, unrecognized; it occurred in some work of Mr. G. H. Hardy's on which I based my argument; and I was really concerned, not so much with the proof that every aggregate can be well-ordered, as with the proof that the series (W) of ordinal numbers has no type.

The matter becomes simpler to express when we consider *classes* instead of *series*. My contention, then, was that there is no such thing as "the cardinal number of the class of ordinal numbers" seems to represent. But if we adopt, as I adopted, the Frege-Russell definition of the cardinal number of a class u as the class of those classes which are similar to (can be put in a one-one correspondence with) u, there arises a difficulty. The cardinal number of the class w of ordinal numbers is the class of those classes which are similar to w; and this class certainly *exists*, for we can point out at least one member of it, namely, w itself, for w is similar to w. On the other hand, we have reason to deny that there *is* such a class as the cardinal number of w, and most mathematicians express this by saying that the cardinal number in question does not "exist." Of course, the solution of this apparent contradiction is that "the cardinal number of w" is a phrase denoting nothing—there is no such entity as the cardinal number of w. If it *did* denote a class, that class would be existent.

So, in my above-quoted paper, I distinguished between the existence of a class u from the entity of a thing v. The symbol "$\exists u$" was used, following Peano, to denote that u exists, and the symbol "Ev" was used to denote the proposition that v is an entity. The symbol "Ev" was defined by the definition of "not-Ev" as "v is a member of the null-class." Since the null-class has no members, and is defined as the x's satisfying a propositional function, such as x is not identical with x, which is always false, this is a most paradoxical way of stating the case about non-entity,[22] and the paradox results from the assumption that, in some sense, there *is* a v,—that, as MacColl would have said, v has a "symbolical

[22] On printing the above article, Professor Peano wrote to me, on Jan. 1, 1906, as follows: ".... I see the new symbol E, which you do not define symbolically, but the importance of which I believe I have understood....It would be necessary to introduce many kinds of null-class (Λ): Λ_0 = that of the *Formulaire*; Λ_1 = the class of classes, which has no classes; Λ_2 for the classes

existence." But, as Dr. Whitehead and Mr. Russell[23] pertinently remark: "We cannot first assume that there is a certain object, and then proceed to deny that there is such an object." Russell's solution of the difficulty about propositions asserting that "the so-and-so is not an entity" is to reduce all such propositions to a form not involving the assumption that "the so-and-so" is a grammatical subject. "The so-and-so," whether it appears to denote something or not, is an *incomplete* symbol, like the d/dx of mathematics.

* * *

It has, I trust, been not quite without interest to see how the important distinction of *existence* and *entity* in mathematics struggled into clearness. We have seen before[24] that the discussions on "existence" of MM. Poincaré and Couturat were conducted in obscurity. This obscurity was produced by the confusion of the two notions of *existence* and *entity*, and the consequent use of one word to denote both.

When, in a paper published in 1904, I used the badly chosen term "inconsistent" for an aggregate whose cardinal number is a non-entity—"does not exist," I said then—Mr. Russell rightly objected that, given a class u, its cardinal number *must* exist, since u is a member of the class called the cardinal number of u. And yet there was an undoubted difficulty about what I called "inconsistent" classes. We know now that—at any rate when the number of a class is defined logically—it is a delusion that there are such "inconsistent" classes,—they are non-entities. If they *were* entities, their cardinal numbers would "exist."

There is one more thing to be noticed: it is the *entity* of a number that is most important, the proof of its *existence* is less so. In his *Principles* of 1903, Mr. Russell laid great stress on the existence-proofs of numbers and classes of spaces. Let us consider the case of real numbers. A real number is, according to Mr.

of classes; Λ_3 for the classes of classes of classes;.... Λ_n,.... Λ_ω,..... There is the generation of the transfinite numbers, in the principles of logic. There results this rather laughable consequence, that the new philosophers have decomposed *nothing* into a transfinite number of classes!"

[23] *Op. cit.*, p. 69. We may remark here, as I have done in a review of Whitehead and Russell's *Principia* in the *Cambridge Review* for 1911, that the authors (cf. pp. 32, 69, 182, 229) use the *word* "existence" ambiguously; though, of course, there is no ambiguity when the proper technical symbols ($\exists$ and E; E only occurring in a phrase involving incomplete symbols) are used.

[24] *Monist*, Jan. 1910, Vol. XX, pp. 113-116.

Russell, a certain class of rational numbers; its existence can be proved, and one feels satisfied. But a rational number or a negative number, being a relation, does not "exist," and yet one would have thought existence quite as important in these cases as in the case of real numbers.[25] I hope to go more fully into this question on another occasion.

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IDEALISM AS A FORCE.

A MECHANICAL ANALOGY.

In the present state of knowledge the man of intelligence has much difficulty in deciding what course of conduct he should adopt in regard to beliefs and social and religious practice without at the same time violating these principles which he has obtained from science and critical philosophy. Before venturing to suggest exactly what position he should (and eventually must) take up, a little consideration of the importance of the older ideas and their relation to new ones would be advisable. I propose to introduce various mechanical analogies in this sketch, for two reasons. First, because I think they show forth more clearly the nature of the phenomena described, and second, a training in scientific thought soon shows one that mechanical laws pervade the whole universe, mental, moral and physical. I do not use the word "mechanical" in at all a derogatory sense. As a matter of fact, although it seems at first contrary to our ideas of perfection realized by a continuous process of adjustment, the really perfect state is the mechanical one, where each part has a definite and unchanging relation to all the other parts, so that a change in its condition is accompanied by a change in all other parts in accordance with the nature of that mutual relation. Surely this is what is meant by "correspondence with environment," if there is the proviso of stability. All moral philosophers have more or less directly stated that the key to morality is the Golden Rule, "Do as you would be done by," or as K'ung-fu-tze puts it, in one word, "Reciprocity," i. e., mutual bearing upon one another. This condition of mutual bearing is essentially, when complete, a mechanical

[25] Frege (*Grundlagen der Arithmetik*, Breslau, 1884, pp. 114-115) indicated such definitions of all the numbers of analysis as would enable him to prove the existence in every case.

state. Similarly in matters of thought consistency is the great principle, and what is consistency but a mechanically perfect state of balance? As to the mechanical character of physical conditions there can be no question, provided we do not necessarily limit the concept to the Newtonian exposition.

I wish to use frequently the idea of *force*. In natural philosophy a force is that which tends to produce or hinder motion, and it is the characteristic of all natural phenomena that the forces acting on them shall be in a state of *balance*. Whether they are still or moving, this balance exists either in the form of opposed pulls, pushes, stresses or accelerations of mass. It is the criterion in the light of which all mechanical problems may be attacked. I wish to extend this idea of force to matters of thought and ideal, by a definition such as the following: A mental force is that which produces or tends to produce change of thought.

The ever-famous Newton, in studying natural forces, announced three laws of motion. There is no definite proof of these, but we have no experience which contradicts them.

With the suggested psychical analogues these laws are as follows:

1. Any body tends to remain in its condition of rest or motion until acted on by some force.

To extend this to matters of thought we can say:

Any idea (group of concepts) tends to remain in its state of rest or change along certain lines until acted upon by some mental force.

2. Change of motion is proportional to the magnitude of the applied force.

This becomes:

Change of thought is greater or less according to the effective importance of the mental force.

3. To every action there is a reaction, i. e., whenever a force acts upon a body there is called out in that body a force opposed to (and equal to) the first force which manifests itself as internal stress or acceleration of mass.

In mental matters this notion is expressed by the change in thought which takes place as the result of applying mental force, appearing either as a new formation of ideas or a reaction of old ideas on the new mental force.

It must be understood at this point that I do not mean anything

extremely mystical or undiscovered by this term "mental force." I simply give this name to a set of ideas, in the first place external to the mind in question, then received through the ordinary channels of sense, and acting upon the ideas already existing there, either producing resistance or modifying those ideas. The technical word "suggestion" is almost identical in meaning.

The engineer, in the spirit of Newton, takes our above-described three laws into one equivalent, as follows:

Force is the rate of change of motion attached to matter (technically "momentum").

This simply means that wherever and whenever a force acts upon a body it produces a change in its motion, or, *vice versa,* a change in motion is caused by a force.

This can be made the basis of a more sweeping statement which describes mental force thus:

Mental force is the rate of change of thought attached to mind. (Brain-matter is perhaps not to be regarded as the absolute medium of thought, since psychologists regard the latter as contemporaneous with, but not necessarily the same as, change in cerebral substance).

Idealism I wish to describe as a particular type of mental force proceeding in the first place from some external source, and then by its action on different minds in accordance with the above laws and by the reactions of such minds on physical and moral actions, producing an effect tending to the realization of certain progressive states which are for the time being regarded as perfect.

In the light of this conception all religions are forms of idealism.

If we examine any religion from its commencement we usually find some such development as this:

1. Absorption by a master mind (the founder) of certain older ideals, the mutual reactions of which together with the mental contition induced in him by his surroundings (physical and social) produce a new system with one central ideal.

2. This result in many cases is accompanied by very severe mental strain, and in some cases by nervous disease (cf. Mohammed who is believed to have suffered from epilepsy) after which this ideal takes the leading part in his thought and life (monoideism).

3. The ideal now works through him to the minds of certain followers or disciples who receive it according to their previous

training and heredity, and so is formed a circle of minds in which the ideal circulates for a time, gaining an ever increasing potential.

4. The widening of the circle and frequently the loss by decease of the founder, causes the ideal to cease its original evolution and take on certain new features according to the reactions in the minds of its various adherents. Hence we have lesser circles forming, to which certain new phases have more and more relation, until there is a schism of the original community and the most energetic minds found sects.

5. These sections expand or not according as the ideal is resisted or absorbed by the further minds upon which it acts, and we may finally have a large community with the ideal (usually much modified by reaction) controlling and connecting the units. This arrangement persists until external ideas of a different kind or internal resistances destroy its energy and it is replaced by other ideals or a great modification of the old one.

The mechanical analogy to the action of external forces on matter already possessing kinetic energy is so obvious if the lines previously indicated are followed, that I will not trace out each link of the chain, but merely point out the steps in which we draw a comparison.

1. Composition (i. e., combining together) of various forces (ideals) in one point (mind) which possesses considerable freedom (enthusiasm).

2. Acceleration in this point (mind) under the resultant force (new ideal) finally acting on other bodies (minds) in a greater or less degree according to their condition of stability (environment).

3. Composition of the forces in these individual bodies (minds) resulting in a balanced but unstable system (idealist community).

4. Splitting up of systems into smaller systems (sects) balanced in themselves with moderately high stability (sects) and balanced as a whole (unstably) as a general system (national religion).

5. Modification of system by new forces (ideals) finally resulting in a new system (religion).

At this point it is necessary to discuss the importance of idealism in its effect on the social life. Once a definite ideal or system of ideals has become established among a set of minds it acts as a "superhuman" power (not in the accepted sense of "supernatural" but as the simple result of evolution) whose magnitude is the resultant of the various forces which it has impressed on individual

minds and whose direction (i. e., tendency to progress or degenerate) is determined by the manner in which it has combined with the mental forces previously impressed on these minds.

We see then that it has a definite (but fluctuating) value, a more or less constant direction (for the time) and it is attached to a certain number of unit minds.

It may be compared with the constitution of the atom in which there are a number of electrons each possessing a peculiar resultant motion of its own but at the same time coordinating with other electrons to confer on the atom as a whole certain dynamic properties which manifest themselves as polarity or chemical attraction, which, although the equivalent of the electronic energy, are different in kind.

Similarly our ideal may be attached to a large number of minds of varying caliber, force and direction, but as a whole organism the system will be possessed of properties differing from those of its units.

Such a force as this centered in a community constitutes a divine being controlling and working through its members, just as according to modern psychology, the soul is a centering of nervous energy. The Christian church in which the members are said to belong to the mystical body of Christ exemplifies this. The whole of the church is, so long as homogeneity prevails, a force whose magnitude is the resultant of the mental and moral efforts of the units. These efforts may be distinct in kind, amount and object, but nevertheless on the whole they are cumulative and there is a resultant which may be well called the living Christ, for it is an intelligent force realizing within itself to some extent the ideal which the master-mind of Jesus impressed on his disciples to such a degree as their capacities permitted.

In this way the doctrines of salvation (i. e., separation from anti-Christian community and ideals) and grace (impression of idealism according to capacity for receiving it) become explicable and even reasonable. Of this more later.

I am of course aware that I at once lay myself open to severe criticism from the adherents of all faiths who conceive their deity to be omnipotent and omniscient. To this notion I would say that such a force as described above has within itself the means of doing and knowing all those things which come within the ken of the units, and that further it combines with the resultant forces of the

universe, being either decreased or increased in effect according as it is opposed to or in line with such world forces. So long as a religion progresses (apart from the consideration of certain artificial conditions such as politics) it must be to some extent in conformity with the laws of the universe, known and unknown. So soon as it directly opposes those laws (still subject however to certain sociological factors) it must degenerate. The gods of a religion live and die with it, their energy appearing in other faiths after reaction has taken place in the minds of the interregnum. The only case in which they (or he) are immortal is when they are definitely identified with some permanent force in the universe so that the mental force runs contemporaneously with a natural one, each producing proportionate effects on mind and matter. It is from this cause that Judaism has ensured its immortality. About the time of the Captivity it definitely connected its tribal deity Yahweh not only with the ideal of *tzedek* (righteousness) but with that unitary world-power which under various names (such as "the eternal energy") all philosophers and scientists recognize, with or without moral attributes. This element of permanence has been transmitted to Christianity and Islam so that these three are probably the most stable of all faiths. It does not however necessarily follow that because the force survives, the attachment of the community to the ideal force will also survive. Its energy may be transferred to other minds, possibly in other forms, but practically never losing all connection with the primal natural force with which it has been associated.

In order that the idealism of a community shall have a permanent effect it is necessary:

1. That there should be a continual supply of mental energy on the part of unit minds;

2. That the individual energies shall be so directed generally and of such amount that there always is an external resultant producing progress by its reaction on the minds of both the units of the community and those outside of the community.

In order to assure the first condition some definite "cult" is required, which by the repetition of various practices concentrates the mind on the ideal tending to develop its realization in that mind and directing the energy of the mind to that end, both within and without.

In the second condition it is essential that certain agreements

concerning the ideal shall be established, so that the energies put forth are not contrary in tendency. This is the foundation of dogma, which states as far as possible the ideal in words and symbols, which produce in the various minds a more or less homogeneous conception of the ideal.

Further, it is necessary in order that the mental forces shall not equilibriate, that all the members of the community shall, as far as practicable within the limits of the competition necessitated by the law of selection and survival, support one another, so that the mutual stress between them is minimized and the external resultant increased.

To return to our electron analogy, if electrons move at right angles to the general path, collisions will occur which reduce the external force exerted by the atom, and if sufficiently numerous may be conceived quite to destroy that force and even disintegrate the atom. (Cf. "The house divided against itself.")

This necessity for internal balance gives rise to ethics, which is summarized by the Golden Rule.

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CLASSICAL CONFUCIANISM.

Sinology has so far not yet passed the stage of crude and amateurish translation. No interpretative work worthy of serious consideration has yet appeared. Mr. Miles Menander Dawson's recently published book, *The Ethics of Confucius: The Sayings of the Master and his Disciples upon the Conduct of the "Superior Man,"*[1] is an attempt in the direction of interpreting Confucianism to the West. We congratulate him on his highly successful exposition of one of the greatest ethical systems of the world. His work has at least met a need which has long been felt by all who desire to bring about a better understanding of Chinese civilization in the occidental world. For ever since the days of Marshman and Legge the true meaning of Confucianism has been lying hidden in those painstaking but unfortunately too expensive and out-of-print translations; and the general public have long had to swallow what superficial and biased writers are pleased to call "Confucianism." Mr. Dawson's book is based entirely on Legge's translation of *The*

[1] New York, G. P. Putnam's Sons. Pp. xviii, 305. Price, $1.50 net.

Chinese Classics, and he has so classified and arranged his material that the reader can easily comprehend what Confucius and the early Confucians actually said on the various fundamental problems of life.

This book has many notable merits. First, the handling of the immense quantity of material is excellent. The work is divided into seven chapters: I. What Constitutes the Superior Man; II. Self-Development; III. General Human Relations; IV. The family; V. The State; VI. Cultivation of the Fine Arts; VII. Universal Relations. Mr. Dawson has seized upon a very important point in Confucianism when he arranges his book in accordance with the scheme of *The Great Learning.* For the Confucian ethics is essentially a system of human relations: all extension of knowledge contributes to the cultivation of individual conduct, and from the individual there radiate the relationships of the family, the state and the world.

Secondly, the illustrative quotations from the Confucian classics are, with a few exceptions, very well chosen. The quotations are all accompanied by the name of the book, the number of chapter, paragraph and verse. The carefulness and patience with which the numerous passages are selected and classified, certainly commands our admiration. The index appended to the book also enhances its usefulness.

Thirdly, the first two chapters in particular constitute the best portion of the book. In these chapters Mr. Dawson sets forth the Confucian ideal man, "the Superior Man," which forms the subtitle of the book. The Superior Man, which can be more literally translated as "the lordly man" or better still as "the gentleman," is quite different from the dianoetic man of the Greeks; neither does he aspire to the Nirvanic life of Buddhism, nor aim at the attainment of a union with God, which forms the ideal of Christianity. The Confucian ideal is simply a life made ever nobler and richer by individual reticence and by a conscious adoption as one's own of the social moral institutions which constitute the *li* (translated "rules of propriety") or what the Hegelians call *Sittlichkeit.* In expounding these basic elements of Confucianism Mr. Dawson has exhibited a high degree of clarity of exposition and richness of illustration.

Lastly, we believe that the greatest merit of the book lies in its objectivity, by which is meant the impartiality and disinterestedness with which the author expounds the Confucian doctrines.

Mr. Dawson has no desire to prove that Confucianism is inferior to any particular ethical or religious system, nor does he wish to proselyte his readers into Confucianism. He simply presents to us what the great Confucians thought and taught concerning the multifarious complexities of life and conduct. He speaks of concubinage with the same calmness with which he discusses the Confucian conception of the state.

It is natural that an undertaking of this kind by one who has no access to the original texts cannot be entirely free from occasional errors. Numerous unimportant mistakes may be pointed out at random. For example: (1) on page xiii, the name of Confucius appears twice as *Kung Chin,* which should be *Kung Chiu*; (2) on page xiv, *Chun Chin* should read *Chun Chiu*; (3) on page xvi, it is wrong to include the *Hsiao King* instead of the *Chun Chiu* in the Five Classics; and (4) on the same page "Pan Ku" and *The History of Han Dynasty* are mentioned as two separate works; whereas, as a matter of fact, Pan Ku is the author of *The History of Han Dynasty.*

Of errors of a more serious nature we find at least three. In the first place, the title, "The Ethics of Confucius," is not correct. It is as if a compilation of the ethical theories contained in the works of Plato, Aristotle and Theophrastus were to be called "The Ethics of Socrates." Mr. Dawson's book deals with the ethics, not of Confucius alone, but of what we may call classical Confucianism. For it is almost needless to point out that many of the Confucian classics, like the *Shu King* and the *Shi King,* deal with historical periods long before Confucius; while others, like the *Book of Mencius* and the *Li Ki,* came long after the death of Confucius. Book III of the *Li Ki,* for example, was compiled in the second century B. C.

In the second place, Mr. Dawson has at times misinterpreted the meaning of certain passages. Take this illustration:

> "The scholar keeps himself free from all stain" (*Li Ki,* xxxviii, 15). The Master said, "Refusing to surrender their wills or to submit to any taint to their persons; such, I think, were Pih-E and Shuh-Tse" (*Analects,* xviii, 8).

"These two passages," says Mr. Dawson, "illustrate the sage's insistence upon sexual continence, among other virtues." Now the word "stain" in the first quotation has no reference to sexual relations. Nor does the phrase "taint to their persons" in the second quo-

tation mean sexual immorality. The story of Pih-E and Shuh-Tse (or Po-I and Shu-Chi), who abandoned their hereditary kingdom and retired into obscurity, and who, when the Chou Dynasty was founded, died of hunger rather than live under the new dynasty,—this story is well known to every Chinese, and is given in a note in Legge's translation (v. 22).

In the third place, Mr. Dawson has on several occasions taken a passage quite apart from its immediate and inseparable context, thus losing the meaning that was intended. An example of this kind is found on page 248:

> "When good government prevails in the empire, ceremonies, music and punitive military expeditions proceed from the emperor" (*Analects,* xvi, 2).

This passage Mr. Dawson takes as "suggesting that wise patronage and encouragement of art by the government which has distinguished the most enlightened governments of ancient and modern times." Now this passage cannot be taken apart from its context. Here is the context:

> "When good government prevails in the empire, ceremonies, music, and punitive military expeditions proceed from the emperor. When bad government prevails, these things proceed from the princes. When these things proceed from the princes, rarely can the empire maintain itself more than ten generations."[2]

Here we can easily see that the point of emphasis in this passage is from what source these institutions should derive their authority. The passage no more illustrates the wise patronage of art than it illustrates the encouragement of punitive expeditions.

It must be pointed out, however, that such errors are very rare in the entire work. On the whole, Mr. Dawson's book may be recommended to all students of Chinese philosophy and religion as an excellent exposition of classical Confucianism.

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[2] This is my translation. Legge's rendering is not correct.